

SPECTRAL STABILITY ANALYSIS FOR SPECIAL SOLUTIONS OF SECOND ORDER IN TIME PDE'S: THE HIGHER DIMENSIONAL CASE

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ABSTRACT. We develop a general theory to treat the linear stability of certain special solutions of second order in time evolutionary PDE. We apply these results to standing waves of the following problems: the Klein-Gordon equation, for which we consider both ground states and certain excited states, the Klein-Gordon-Zakharov system and the beam equation. We also discuss applications to excited states for the Klein-Gordon model as well as multidimensional traveling waves (not necessarily homoclinic to zero) for general second order equations of this type. In all cases, our abstract results provide a complete characterization of the linear stability of such solutions.

1. INTRODUCTION

In this article, we consider second order in time evolutionary equations/systems in the form

$$(1) \quad u_{tt} + \mathcal{L}u - f(|u|^2)u = 0 \quad (t, x) \in \mathbf{R}_+^1 \times \mathbf{R}^d \quad \text{or} \quad (t, x) \in \mathbf{R}_+^1 \times [-L, L]^d,$$

where the nonlinearity $f : \mathbf{R}^1 \rightarrow \mathbf{R}^1$ and the (unbounded) self-adjoint linear differential operator \mathcal{L} are to be made precise in each concrete example.

We will be interested in the linear stability of various special solutions of nonlinear PDEs. In order to focus the discussion, we start with the most natural example, which fits our framework - the standing wave solutions of (1). These objects have been studied extensively in the last thirty years and many methods have been developed to study their stability properties. We would like to use them as a starting example, in order to motivate our approach and the abstract results that will address these issues.

Going back to the standing wave solutions, these are solutions in the form $u(t, x) = e^{i\omega t}\varphi_\omega(x)$, where $\omega \in \mathbf{R}^1$ and φ_ω is real-valued. Such solutions satisfy the stationary equation

$$(2) \quad \mathcal{L}\varphi - \omega^2\varphi - f(\varphi^2)\varphi = 0$$

In order to ease into the notion of linear stability, which will be the main focus, let us consider the linearization of the equation (1). To that end, let $u = e^{i\omega t}(\varphi_\omega(x) + v(t, x))$ and plug it into (1). This is of course still a nonlinear equation for v . Assuming that v is small, it is reasonable to ignore all the terms in the form $O(v^2)$. We arrive at the following *linear* equation for v

$$(3) \quad v_{tt} + 2i\omega v_t - \omega^2 v + \mathcal{L}v - f(\varphi^2)v - 2\varphi^2 f'(\varphi^2)\Re v = 0.$$

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Separating the real and imaginary part, with the assignment $\mathbf{v} = (\Re v, \Im v)$ yields the following system for \mathbf{v} ,

$$(4) \quad \mathbf{v}_{tt} + 2\omega J \mathbf{v}_t + \mathcal{H} \mathbf{v} = 0,$$

where

$$\begin{aligned} J &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}, \\ L_+ &= \mathcal{L} - \omega^2 - f(\varphi^2) - 2\varphi^2 f'(\varphi^2) \\ L_- &= \mathcal{L} - \omega^2 - f(\varphi^2). \end{aligned}$$

Note that if the function f is increasing, the self-adjoint operators satisfy $L_- \geq L_+$.

We would like to point out that the standing wave solutions are by no means the only example that fits our theory. As we shall see below, our results are applicable to multi-dimensional traveling waves as well as standing-traveling waves. Additional applications include the recent work by the first named author, [40], where the stability of subsonic traveling waves for the Benney-Luke model is completely characterized.

In order to give a definition of linear stability, we assume that the linear system (4) has global solutions for all sufficiently smooth and decaying data. This is of course equivalent to saying that the operator

$$\tilde{\mathcal{H}} = \begin{pmatrix} 0 & 1 \\ -\mathcal{H} & -2\omega J \end{pmatrix}$$

generates a C_0 semi-group on appropriate spaces, but this is sometimes hard to verify in concrete examples. In any case, under this assumption, we say that the standing wave $e^{i\omega t} \varphi_\omega$ is linearly stable, if the solution to the linear system (4) satisfies $\lim_{t \rightarrow \infty} e^{-\delta t} \|\mathbf{v}(t)\| = 0$ for any $\delta > 0$ and for a dense set of appropriate initial data.

Similarly, we say that the system is spectrally stable, if the spectrum of $\tilde{\mathcal{H}}$ lies in the closed left half-space. That is $\sigma(\tilde{\mathcal{H}}) \subseteq \{z : \Re z \leq 0\}$. Note that under the standard assumption that $\tilde{\mathcal{H}}$ generates a C_0 semi-group, linear stability implies spectral stability, but not vice versa. Under some natural extra assumptions however (which guarantee the validity of the so-called spectral mapping theorem), the spectral stability is indeed equivalent to linear stability. We will not explore this connection any further, but the interested reader can consult the book [6], the excellent survey paper [7] as well as [32] and [9].

One also has the related notion of nonlinear (orbital) stability. Basically, this means that if one starts close to the standing wave, then the solution will stay close to the wave, modulo the invariance of the system under consideration. The notion of asymptotic stability is the strongest of all and it requires the difference between the two close solutions (modulo the invariance) to go to zero as time goes to infinity. We will not pursue these issues here, except to mention that establishing linearized stability is a prerequisite for asymptotic stability results and thus, the results in this paper should be viewed as an important step toward accomplishing such a goal.

1.1. Examples. We consider the following models - the Klein-Gordon equation, the Klein-Gordon-Zakharov system and the beam equation in the whole space contexts, although the methods developed herein will be certainly useful for other examples and/or periodic domains. Also, we mainly consider standing wave solutions, although towards the end of the discussion, we offer some ideas on how to obtain stability/instability results for multidimensional traveling waves as well, see Section 2.5.

All of these models have been the subject of an intensive investigation in the last thirty years, with the majority of the results concerning orbital stability/instability. This was partly due to

the the versatility of the general theory, developed by Grillakis-Shatah-Strauss for such equations/systems. We provide more specific references to these studies after our theorems, which once again concern *the linear stability* of their special solutions.

We begin with some basic setup, which has dual purpose: on one hand, it will motivate our approach to the problem at hand, and on the other, it will set the stage for the proofs in the subsequent sections. We start with the Klein-Gordon model.

1.1.1. *Klein-Gordon equation: ground states.* Consider

$$(5) \quad u_{tt} - \Delta u + u - |u|^{p-1}u = 0$$

This clearly fits the profile (1), where the operator $\mathcal{L} := -\Delta + 1$. It is well-known that in this case, the corresponding operator $\tilde{\mathcal{H}}$ generates a C_0 semigroup, see [30]. Let us consider some general properties of the operators \mathcal{H}, L_{\pm} , depending on the type of solutions φ_{ω} that one encounters. Observe that, if we consider only decaying solutions of (2), we can conclude that $\sigma_{a.c.}(L_{\pm}) \subset [1 - \omega^2, \infty)$ by Weyl's theorem. Note that by (2), $L_-[\varphi] = 0$. Moreover, if φ does not change sign (say, we take it to be positive), it follows by Sturm-Liouville's theory that $L_- \geq 0$ and 0 is a simple eigenvalue. That is $\sigma(L_-) \subset [0, \infty)$ and $L_-|_{\{\varphi\}^{\perp}} \geq \kappa^2 > 0$.

In addition, differentiating (2) with respect to the spatial variables produces the identity $L_+[\nabla_x \varphi] = 0$, whence $\text{Ker}[L_+]$ is at least d dimensional, with eigenfunctions $\frac{\partial \varphi}{\partial x_j} : j = 1, \dots, d$. Usually, $\text{Ker}[L_+] = \text{span}[\frac{\partial \varphi}{\partial x_j} : j = 1, \dots, d]$, but this is by no means automatic. Note also that

$$\langle L_+[\varphi], \varphi \rangle = -(p-1) \int \varphi^{p+1}(x) dx < 0,$$

thus guaranteeing the presence of a negative point spectrum for L_+ . In the seminal papers by Shatah, [35] and Weinstein, [41], most of the spectral properties for the operators L_{\pm} were established. The full and complete analysis of the spectral properties of L_{\pm} was subsequently given by Kwong in [24]. We also recommend the excellent paper [4] for a more contemporary approach to these facts.

To summarize the known results in the case of power nonlinearities, for $p \in (1, p_{\max})$,

$$p_{\max} = \begin{cases} 1 + \frac{4}{d-2} & d \geq 3 \\ \infty & d = 1, 2 \end{cases}$$

we have $L_- \geq 0$, $L_-[\varphi] = 0$, $L_-|_{\{\varphi\}^{\perp}} \geq \kappa^2 > 0$, while $L_+ : \text{Ker}[L_+] = \text{span}[\frac{\partial \varphi}{\partial x_j} : j = 1, \dots, d]$, with single simple negative eigenvalue, $L_+[\phi] = -\sigma_0^2 \phi$ and $L_+|_{\{\phi, \nabla \varphi\}^{\perp}} \geq \kappa^2 > 0$

We now turn our attention to the problem for excited states of the Klein-Gordon model.

1.1.2. *Klein-Gordon: excited states (vortices) in two dimensions.* Besides the ground states solutions, whose properties were described in Section 1.1.1 above, there are numerous other "excited" solutions of (5). For example, P.L. Lions, [27] has constructed stationary solutions in even dimensions¹ $d = 2k$ in the form

$$\phi(r_1, \dots, r_k) e^{i(m_1 \theta_1 + \dots + m_k \theta_k)}, m \in \mathbf{Z}^k.$$

where $(r_j, \theta_j), j = 1, \dots, k$ are the polar variables corresponding to (x_{2j-1}, x_{2j}) . In the case of two spatial dimensions, this work has been extended by Iaia and Warchal, [15], who have shown that there are infinitely many solutions in the form $\phi_{m,k,p}(r) e^{im\theta}$. More precisely, these satisfy

$$(6) \quad -\phi''(r) - \frac{1}{r} \phi'(r) + \frac{m^2}{r^2} \phi(r) + \phi(r) - |\phi(r)|^{p-1} \phi = 0.$$

¹and similar in odd dimensions, which we do not consider herein

where the equation (6) is supplied by the natural boundary conditions $\lim_{r \rightarrow 0^+} r^{-m} \phi(r) = 0$, $\lim_{r \rightarrow 0^+} r^{-m+1} \phi'(r) = m\alpha$ for some $\alpha \geq 0$ and k stands for the number of zeros of $\phi_{m,k,p}(r)$. In a subsequent work, Mizumachi, [29] has shown the uniqueness of the positive solutions of (6) (i.e. for $k = 0$) and in addition, he has shown the orbital stability for $1 < p < 3$ (and instability for $p > 3$) of the standing waves $e^{i(\omega t + m\theta)} \phi_{m,0,p}(r)$, where these are understood as time periodic solutions to the Schrödinger equation and the perturbations are taken to be in the form $e^{i(\omega t + m\theta)} z(r)$. We encourage the reader to consult the excellent paper, [4], where these and other results are reviewed in full detail, including a number of high-precision numerical verifications thereof.

Note that one can construct in the same manner solutions of (5) in the form $e^{i(\omega t + m\theta)} \phi_{\omega;m,0,p}(r)$, where

$$\phi_{\omega;m,0,p}(r) := (1 - \omega^2)^{\frac{1}{p-1}} e^{i(\omega t + m\theta)} \phi_{m,0,p}(\sqrt{1 - \omega^2} r), \omega \in (-1, 1),$$

where $\phi_{m,0,p}(\cdot)$ is the unique positive solution of (6). Linearizing such solutions with respect to the ansatz $u = e^{i(\omega t + m\theta)} (\phi_{\omega;m,0,p}(r) + v(r))$ yields the linearized equation (4), where the operators L_{\pm} act on the subspace of radial functions via the formulas

$$\begin{aligned} L_+ &= -\Delta + (1 - \omega^2) - p\phi_{\omega}^{p-1} = -\partial_{rr} - \frac{1}{r}\partial_r + \frac{m^2}{r^2} + (1 - \omega^2) - p\phi_{\omega}^{p-1} \\ L_- &= -\Delta + (1 - \omega^2) - \phi_{\omega}^{p-1} = -\partial_{rr} - \frac{1}{r}\partial_r + \frac{m^2}{r^2} + (1 - \omega^2) - \phi_{\omega}^{p-1} \end{aligned}$$

Mizumachi has showed (see Proposition 3.1, [29]) that on the subspace of radial functions, L_+ has exactly one simple negative eigenvalue, say $-\sigma^2$ and $\sigma(L_+) \setminus \{-\sigma^2\} \subset (0, \infty)$, while 0 is a simple eigenvalue of L_- (which as evidenced by (6) comes with the positive eigenfunction ϕ_{ω}) and $\sigma(L_-) \setminus \{0\} \subset (0, \infty)$. That is, the matrix self-adjoint operator $\mathcal{H} = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}$, has a simple negative eigenvalue $-\sigma^2$, a simple eigenvalue at 0 and the rest of the spectrum is contained in (λ_0, ∞) for some $\lambda_0 > 0$.

1.1.3. *Klein-Gordon-Zakharov system.* Consider the Klein-Gordon-Zakharov system

$$(7) \quad \begin{cases} u_{tt} - u_{xx} + u + nu = 0 & (t, x) \in \mathbf{R}_+^1 \times \mathbf{R}^1 \\ n_{tt} - n_{xx} - \frac{1}{2}(|u|^2)_{xx} = 0, \end{cases}$$

which describes the interaction of a Langmuir wave and an ion acoustic wave in a plasma. We now describe a two parameter family of standing-traveling wave solutions, which were derived in [5]. More precisely, let $\omega, c : \omega^2 + c^2 < 1, q = \frac{\omega c}{1 - c^2}$. Then, the pair $u(t, x), n(t, x)$ given by

$$\begin{cases} u(t, x) &= e^{-i\omega t + iq(x - ct)} \varphi_{\omega, c}(x - ct) \\ n(t, x) &= \psi_{\omega, c}(x - ct) \end{cases}$$

where

$$\begin{aligned} \varphi_{\omega, c}(y) &= \sqrt{2(1 - \omega^2 - c^2)} \operatorname{sech} \left(\frac{\sqrt{1 - \omega^2 - c^2}}{1 - c^2} y \right) \\ \psi_{\omega, c}(y) &= -\frac{(1 - \omega^2 - c^2)}{1 - c^2} \operatorname{sech}^2 \left(\frac{\sqrt{1 - \omega^2 - c^2}}{1 - c^2} y \right) \end{aligned}$$

In the case of standing waves, we set $c = 0$ (and hence $q = 0$) and denote $\varphi_\omega := \varphi_{\omega,0}, \psi_\omega := \psi_{\omega,0}$. Linearize the problem around the special solution $(e^{-i\omega t}\varphi_\omega, \psi_\omega)$, that is take the ansatz

$$\begin{aligned} u(t, x) &= e^{-i\omega t}(\varphi_\omega(x) + v(t, x)), \\ n(t, x) &= \psi_\omega(x) + w(x), \end{aligned}$$

where v is complex valued, but w is a real-valued function. After dropping all terms quadratic in v, w , we obtain the following linear system

$$(8) \quad \begin{cases} v_{tt} - v_{xx} - 2i\omega v_t + (1 - \omega^2)v + \psi_\omega v + \varphi_\omega w = 0 \\ w_{tt} - w_{xx} - (\varphi_\omega \Re v)_{xx} = 0. \end{cases}$$

At this point, in order to reduce our linearized problem to (4), we introduce a new unknown function $z : w = z_x$. Clearly, if we show instability in the coordinates z , we would have shown instability in the coordinates of w , by simply taking $w = z_x$. While the reverse implication is not so straightforward, it turns out that one can produce instability in the z coordinates, once we have instability in the w coordinates. This is all discussed in Section 6. In any case, the last equation becomes, in terms of z ,

$$(z_{tt} - z_{xx} - (\varphi_\omega \Re v)_x)_x = 0$$

Since we are looking for z with decay at $\pm\infty$ and since φ_ω also decays at $\pm\infty$, we integrate in x this last equation to get

$$z_{tt} - z_{xx} - (\varphi_\omega \Re v)_x = 0$$

Next, we introduce the real and imaginary part of v as new variables $v = \Re v + i\Im v =: \xi + i\eta$. We now have the following linearize system for the triple (ξ, z, η) ,

$$(9) \quad \begin{cases} \xi_{tt} + 2\omega\eta_t - \xi_{xx} + (1 - \omega^2)\xi + \psi_\omega\xi + \varphi_\omega z_x = 0 \\ z_{tt} - z_{xx} - (\varphi_\omega \xi)_x = 0 \\ \eta_{tt} - 2\omega\xi_t - \eta_{xx} + (1 - \omega^2)\eta + \psi_\omega\eta = 0 \end{cases}$$

This last expression is in the form (4), namely

$$(10) \quad \begin{pmatrix} \xi \\ z \\ \eta \end{pmatrix}_{tt} + 2\omega J \begin{pmatrix} \xi \\ z \\ \eta \end{pmatrix}_t + \mathcal{H} \begin{pmatrix} \xi \\ z \\ \eta \end{pmatrix} = 0$$

where

$$\begin{aligned} J &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \mathcal{H} = \begin{pmatrix} H & A & 0 \\ A^* & H_0 & 0 \\ 0 & 0 & H \end{pmatrix} \\ H &= -\partial_x^2 + (1 - \omega^2) + \psi_\omega; H_0 = -\partial_x^2 \\ Az &= \varphi_\omega z_x, A^* z = -(\varphi_\omega z)_x \end{aligned}$$

In fact, in order to emphasize the analogy with the previous case, we set H_\pm , so that

$$\mathcal{H} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \text{ that is}$$

$$(11) \quad H_+ = \begin{pmatrix} H & A \\ A^* & H_0 \end{pmatrix}, H_- = H.$$

1.2. Main abstract result. Before we state our results, we shall make a number of key assumptions, which are motivated by the Klein-Gordon example that we have considered.

First, there exists a negative eigenvalue for L_+ and hence for \mathcal{H} , say with an eigenvector $\begin{pmatrix} \phi \\ 0 \end{pmatrix}$, $\|\phi\| = 1$. It is helpful to assume, that $L^2 = X^+ \oplus X^-$, so that \mathcal{H} acts invariantly on both X^\pm and $J : X^\pm \rightarrow X^\mp$. Moreover, both \mathcal{H}, J map real-valued functions into real valued functions. Finally, we shall require that J is \mathcal{H} -bounded in the sense that for all large enough τ , $J(\mathcal{H} + \tau)^{-1}$ is a bounded operator.

Next, we have a number of eigenvectors in the kernel of \mathcal{H} , denote them by $\psi_0 = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in X^-$ for $\begin{pmatrix} 0 & 0 \\ 0 & L_- \end{pmatrix}$ and $\psi_1, \dots, \psi_n \in X^+$ for $\begin{pmatrix} L_+ & 0 \\ 0 & 0 \end{pmatrix}$. We shall assume that they are all orthogonal to each other and L^2 normalized. Note here that we do not require that $n = \dim(\ker(L_-)) = d$ (as is the case in most applications, but it may be difficult to check rigorously), but we do require that $L_- \geq 0$ and $L_-|_{\{\varphi\}^\perp} \geq \kappa^2 > 0$.

In addition, observe that $J\phi \perp \psi_j, j = 1, \dots, n$, since they belong to X^- and X^+ respectively. On the other hand that $J\phi$ is not necessarily orthogonal to ψ_0 . In fact, in the case of ground states for the Klein-Gordon equation, we have $\langle J\phi, \psi_0 \rangle = -\langle \phi, \varphi \rangle < 0$, since both φ, ϕ are positive functions. We collect our assumptions in the following

$$(12) \quad \overline{\mathcal{H}u} = \mathcal{H}\bar{u}, \quad \mathcal{H} : X^\pm \rightarrow X^\pm, \quad \mathcal{H}^* = \mathcal{H},$$

$$(13) \quad \overline{Ju} = J\bar{u}, \quad J : X^\pm \rightarrow X^\mp, \quad J^* = -J, \quad \forall \tau \gg 1 : J(H + \tau)^{-1} \in B(L^2)$$

Note that as an immediate consequence of $J^* = -J$, we get that for all real-valued functions x , $\langle Jx, x \rangle = 0$. In addition to (12), (13), we assume the following for the eigenvectors of \mathcal{H}

$$(14) \quad \begin{cases} \mathcal{H}\phi = -\delta^2\phi, \mathcal{H}|_{\{\phi\}^\perp} \geq 0; \sigma_{a.c.}(\mathcal{H}) \subset [\kappa^2, \infty), \kappa > 0 \\ \text{Ker}[\mathcal{H}] = \text{span}[\psi_0, \psi_1, \dots, \psi_n], \|\psi_j\| = 1, j = 0, \dots, n \\ \psi_0 \in X^-; \{\phi, \psi_1, \dots, \psi_n\} \in X^+; \langle \psi_i, \psi_j \rangle = 0, j \neq k; \end{cases}$$

Note that in our assumptions, we require that the spectrum of \mathcal{H} has a spectral gap from zero. This is always problematic in other index counting theories and one has to resort to approximation arguments to deal with this issue, whenever it arises. We can however relax that assumption to allow $\sigma_{a.c.}(\mathcal{H}) = [0, \infty)$ under some extra requirements, see Theorem 1.

Next, we require

$$(15) \quad \langle \psi_j, J\psi_0 \rangle = 0, \quad j = 1, \dots, n.$$

This will guarantee that the operator $\mathcal{H}^{-1}[J\psi_0]$ is well defined since $J\psi_0 \perp \text{Ker}[\mathcal{H}]$. Note that (15) does not follow from the other assumptions (say from some orthogonality argument), since both vectors belong to X^+ and in fact, this needs to be verified in each specific instance. **On the other hand, we do not have an example when (15) is not satisfied, but we certainly are not aware of any general result in this direction.** For example, in the case of ground states for KG, we have $\langle \psi_j, J\psi_0 \rangle = -\langle \partial_j \varphi, \varphi \rangle = 0$.

Based on the concrete problem (3), we consider the following more general problem

$$(16) \quad \lambda^2\psi + 2z\lambda J\psi + \mathcal{H}\psi = 0.$$

Namely, given a real number $z \neq 0$, and operators J and \mathcal{H} as above, we are asking whether there is a solution (λ, ψ) to (16). More precisely, we have the following

Definition 1. We say that the pencil associated to (z, J, \mathcal{H}) is unstable, if there exists $\lambda : \Re \lambda > 0$, and a function $\psi \in D(\mathcal{H}) \subset L^2$, so that (16) is satisfied². In such a case, λ is called an unstable eigenvalue for the pencil. Otherwise, we say that the pencil (z, J, \mathcal{H}) is stable.

The following theorem is the main result of this paper.

Theorem 1. Assume (12), (13), (14) and (15) hold. Then,

- the pencil (z, J, H) is unstable if

$$\langle \mathcal{H}^{-1}[J\psi_0], J\psi_0 \rangle \geq 0$$

- (z, J, H) is unstable, if $\langle \mathcal{H}^{-1}[J\psi_0], J\psi_0 \rangle < 0$ and

$$|z| < \frac{1}{2\sqrt{-\langle \mathcal{H}^{-1}[J\psi_0], J\psi_0 \rangle}}.$$

- (z, J, H) is stable, if $\langle \mathcal{H}^{-1}[J\psi_0], J\psi_0 \rangle < 0$

$$|z| \geq \frac{1}{2\sqrt{-\langle \mathcal{H}^{-1}[J\psi_0], J\psi_0 \rangle}}.$$

Remark: We can relax the spectral gap assumption $\sigma_{a.c.}(\mathcal{H}) \subset [\kappa^2, \infty)$ to $\sigma_{a.c.}(\mathcal{H}) \subset [0, \infty)$ by additionally requiring that the operator \mathcal{H} is invertible on the subspace

$$S = \text{span}\{P_{>0}[J\phi], P_{>0}[J\psi_0], P_{>0}[J\psi_1], \dots, P_{>0}[J\psi_n]\},$$

where the projection $P_{>0}$ is defined via $P_{>0}[h] = h - \langle h, \phi \rangle \phi - \sum_{j=0}^n \langle h, \psi_j \rangle \psi_j$. Note that we already have that $S \perp \text{Ker}(\mathcal{H})$, since $P_{>0}$ projects away from $\text{Ker}(\mathcal{H})$. However, if the a.c. spectrum touches the origin, this Fredholm condition, by itself, is insufficient to guarantee the invertibility of \mathcal{H} on this subspace.

In order to state the result precisely, set P_S to be the orthogonal projection onto the $n+2$ dimensional space S . Thus, replacing (14) by

$$(17) \quad \begin{cases} \mathcal{H}\phi = -\delta^2\phi, \mathcal{H}|_{\{\phi\}^\perp} \geq 0; \sigma_{a.c.}(\mathcal{H}) \subset [0, \infty), \\ \text{Ker}[\mathcal{H}] = \text{span}[\psi_0, \psi_1, \dots, \psi_n], \|\psi_j\| = 1, j = 0, \dots, n \\ \psi_0 \in X^-; \{\phi, \psi_1, \dots, \psi_n\} \in X^+; \langle \psi_i, \psi_j \rangle = 0, j \neq k; \\ P_S \mathcal{H} P_S : S \rightarrow S \text{ is invertible} \end{cases}$$

we have that Theorem 1 holds under the assumptions (12), (13), (17) and (15). Morally, if the a.c. spectrum touches the origin, we need to require that the expressions $\langle \mathcal{H}_{>0}^{-1}[Jh_1], [Jh_2] \rangle$ makes sense whenever $h_1, h_2 \in \{\phi, \psi_0, \dots, \psi_n\}$.

Theorem 1 provides a complete characterization of the stability of the pencil (z, J, \mathcal{H}) . We have the following equivalent formulation, which introduces the index $z^*(\mathcal{H})$.

Corollary 1. Under the assumptions of Theorem 1, the pencil (z, J, \mathcal{H}) is stable if and only if

$$|z| \geq z^*(\mathcal{H}) = \begin{cases} +\infty & \langle \mathcal{H}^{-1}[J\psi_0], J\psi_0 \rangle \geq 0 \\ \frac{1}{2\sqrt{-\langle \mathcal{H}^{-1}[J\psi_0], J\psi_0 \rangle}} & \langle \mathcal{H}^{-1}[J\psi_0], J\psi_0 \rangle < 0 \end{cases}$$

There are various works that study in some way the stability of eigenvalue pencils, see for example [1], [28] as well as [31]. We now discuss some more recent and relevant results in the literature and how they compare to Theorem 1. The result is certainly reminiscent to the GSS theory, [12]. In fact, one can formally reduce the quadratic pencil eigenvalue problem (16) to

²Note that the case $z = 0$ is trivial, since the pencil is unstable. In fact, we can pick $\psi := \phi$ and $\lambda := \delta$.

a standard eigenvalue problem appearing in the Grillakis-Shatah-Strauss framework. Namely setting $\mathcal{J} := \begin{pmatrix} 0 & -Id \\ Id & J \end{pmatrix}$, $\mathcal{L} := \begin{pmatrix} \mathcal{H} & 0 \\ 0 & I \end{pmatrix}$ one can recast (16) as

$$\mathcal{J}\mathcal{L}\vec{u} = \lambda\vec{u}.$$

Note also that $\mathcal{J}^* = -\mathcal{J}$, $\mathcal{L}^* = \mathcal{L}$ making it somewhat similar to what one needs to apply the GSS theory. *This approach however will not work, at least for some of the applications that we have in mind.* Indeed, even in the most recent and sophisticated versions, namely the instability index counting formulas in [2, 18, 16, 17], there is a requirement that the operator \mathcal{J} has bounded inverse, at least on a finite co-dimension subspace. Clearly, since, $\mathcal{J}^{-1} = \begin{pmatrix} J & Id \\ -Id & 0 \end{pmatrix}$, it follows that in order for the more classical theories to work, one needs J to be a bounded operator. While this is certainly true in the case of standing waves, it fails to be true for the cases of traveling and standing-traveling waves (because J will contain ∂_x). We should point out that in the recent paper [2], the authors have found a nice way to circumvent this difficulty in the case of *periodic perturbations*, but this does not seem to carry over to whole space problems, roughly due to the appearance of continuous spectrum.

2. APPLICATIONS

We list below a number of applications, which fall under the scope of Theorem 1. Note that for most of them, we are able to verify rigorously the requirements (12), (13), (14), (15). We also present some further possible applications open ended. That is, we generally describe the scheme, under which linear stability may be established, without supplying the full details - this is either because we are unable to verify analytically all conditions in Theorem 1 or because we deal with a very general situation, where we don't even know whether the waves themselves exists (and what their properties are etc.), but some authors will hopefully find useful in their future investigations.

2.1. Klein-Gordon: classical ground states. Our first application regards the linear stability/instability for ground states for the Klein-Gordon equation (5). Before we state our results, let us take the opportunity to review the current state of affairs. First, to the best of our knowledge, Theorem 2 below is the first linear/spectral stability result for ground states of the KG equation. Next, there exists quite an extensive list of papers, dealing with the orbital stability of such waves. We have the following (possibly incomplete) list of results on orbital stability for the Klein-Gordon equation (2).

Regarding orbital instability, Grillakis, [10], [11], (see also Grillakis-Shatah-Strauss, [12]) established sufficient conditions, when one restricts to radial perturbations. Shatah, [35] proved orbital instability, when $d \geq 3$ and $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$. Around the same time, Shatah and Strauss, [36] have shown orbital instability, when $d \geq 3$ and $1 < p < 1 + \frac{4}{d}$, $|\omega| < \omega_{p,d}$, or when $p \geq 1 + \frac{4}{d}$, where $\omega_{p,d}$ is defined in Theorem 2 below.

For orbital stability, Shatah, [34] has shown that it holds for radial perturbations, provided $d \geq 3$ and $1 < p < 1 + \frac{4}{d}$, $\omega_{p,d} < |\omega| < 1$. Our result presents a complete characterization for the linear stability for all values of $p > 1$, all dimensions $d \geq 1$ and all values of $\omega \in (-1, 1)$.

Theorem 2. *Let $p > 1$, $d \geq 1$ and $\varphi_\omega, \omega \in (-1, 1)$ be the unique radial ground state solution of (5). Then*

- *If $1 + \frac{4}{d-2} > p \geq 1 + \frac{4}{d}$, then φ_ω is linearly unstable for all $\omega \in (-1, 1)$.*

- If $1 < p < 1 + \frac{4}{d}$, φ_ω is linearly unstable for

$$0 \leq |\omega| < \sqrt{\frac{p-1}{4-(p-1)(d-1)}} =: \omega_{p,d}.$$

- If $1 < p < 1 + \frac{4}{d}$, φ_ω is linearly stable in the complementary range

$$1 > |\omega| \geq \omega_{p,d}.$$

Note that, if $1 < p < 1 + \frac{4}{d}$, then $\omega_{p,d} \in (0, 1)$.

Note that this result matches exactly the corresponding results for the orbital stability and instability, whenever they exist, possibly with the exception of $\omega = \pm\omega_{p,d}$, where one has linear stability, but it may have secular orbital instability.

Note that the requirement $p < 1 + \frac{4}{d-2}$ is imposed only for the purposes of existence of such unique ground states. It is well-known that such solutions may exist in the range $p \geq 1 + \frac{4}{d-2}$, although they are usually not of finite energy.

2.2. Klein-Gordon: 2D excited states. We now focus our attention to the 2D vortex solutions, described in Section 1.1.2. The result looks almost identical to Theorem 2 for the case $d = 2$.

Theorem 3. *Let $d = 2$, $p > 1$. Consider the unique “excited” standing wave solutions $e^{i(\omega t + m\theta)}\phi_{\omega;m,0,p}$, $\omega \in (-1, 1)$. Then,*

- If $p \geq 3$, these waves are linearly unstable³,
- If $1 < p < 3$, these waves are still linearly unstable, if

$$0 \leq |\omega| < \sqrt{\frac{p-1}{5-p}},$$

- If $1 < p < 3$, these waves are linearly stable, if the perturbations are in the form $e^{i(\omega t + m\theta)}z(r)$, provided

$$1 > |\omega| \geq \sqrt{\frac{p-1}{5-p}}.$$

2.3. Klein-Gordon-Zakharov system: standing waves. Our final application concerns the stability of the standing wave solutions $(e^{-i\omega t}\varphi_\omega, \psi_\omega)$ of the KGZ system. More precisely, we have

Theorem 4. *Let $(e^{-i\omega t}\varphi_\omega, \psi_\omega), \omega \in (-1, 1)$ be the standing wave solution described in (5). These waves are linearly stable if $|\omega| \in [\frac{\sqrt{2}}{2}, 1)$ and unstable for $|\omega| \in [0, \frac{\sqrt{2}}{2})$.*

We should note that the stability results in Theorem 4 match the orbital stability results of Chen, [5]. Note that the orbital instability in the regime $|\omega| < \frac{\sqrt{2}}{2}$ is an open problem, which is likely to follow in a straightforward fashion from the linear instability, established in Theorem 4.

³In fact, the eigenfunction corresponding to the unstable mode has the same equivariant form $e^{i(\omega t + m\theta)}\eta(r)$.

2.4. The beam equation: standing waves. We outline here the results for the beam equation, which have been studied recently by various authors (mainly in the framework of GSS theory for orbital stability/instability). More precisely, we consider the model

$$(18) \quad u_{tt} + \Delta^2 u + u - |u|^{p-1}u = 0,$$

which was studied by Levandosky, [26] (see Section 7). It is shown in this paper that one can find a standing wave solution $e^{i\omega t}\varphi_\omega(|x|)$, $\omega \in (-1, 1)$ of (18), for

$$1 < p < p_{max} = \begin{cases} \infty & d = 1, 2, 3, 4 \\ 1 + \frac{8}{d-4} & d \geq 5 \end{cases}$$

Following the linearization procedure similar to the Klein-Gordon model, we arrive at the form (4), where $\mathcal{H}^{beam} = \begin{pmatrix} L_+^{beam} & 0 \\ 0 & L_-^{beam} \end{pmatrix}$, and the self-adjoint operators L_\pm^{beam} take the form

$$\begin{aligned} L_+^{beam} &= \Delta^2 + (1 - \omega^2) - p\varphi_\omega^{p-1} \\ L_-^{beam} &= \Delta^2 + (1 - \omega^2) - \varphi_\omega^{p-1} \end{aligned}$$

Assuming the spectral information needed for the application of Theorem 1, we have the following

Theorem 5. *Assume that for some $\lambda_0 > 0$*

- L_-^{beam} has a simple eigenvalue at 0, with eigenvector φ_ω , $\sigma(L_-^{beam}) \setminus \{0\} \subset (\lambda_0, \infty)$
- L_+^{beam} has a simple negative eigenvalue $-\sigma_0^2$, $Ker(L_+^{beam}) = span\{\partial_1\varphi_\omega, \dots, \partial_d\varphi_\omega\}$ and $\sigma(L_+^{beam}) \setminus \{-\sigma_0^2, 0\} \subset (\lambda_0, \infty)$

then, the standing wave $e^{i\omega t}\varphi_\omega(|x|)$ is unstable, exactly when $p_{max} > p \geq 1 + \frac{8}{d}$, $\omega \in (-1, 1)$ or $1 < p < 1 + \frac{8}{d}$ and $0 \leq |\omega| < \sqrt{\frac{p-1}{4-(p-1)(\frac{d}{2}-1)}} =: \omega_{p,d}^{beam}$.

Note:

- (1) The spectral assumptions in Theorem 5 have been verified numerically in [8], at least in the one dimensional case.
- (2) The results of Theorem 5 mirror those obtained by Levandosky, [26] for the orbital stability of the same waves⁴. It should be noted however that the formulation of the results in [26] is slightly confusing, as the author states the instability region as $p > 1 + \frac{8}{d-2}$ or $p < 1 + \frac{8}{d-2}$, $|\omega| \leq \omega_{p,d}^{beam}$. However, if $1 + \frac{8}{d} \leq p < 1 + \frac{8}{d-2}$, $\omega_{p,d}^{beam} > 1$, and thus, one still has instability (according to the inequalities listed in [26]), since $|\omega| < 1 \leq \omega_{p,d}$.

2.5. Further applications: multi-dimensional traveling waves. In this section, we present an approach to obtain linear stability/instability results for multidimensional traveling wave solutions. Rather than stating formal theorems, we will organize our presentation as a road map for such results. More precisely, we will be interested in multidimensional traveling waves for Hamiltonian models in the form $u_{tt} + \mathcal{L}u - \mathcal{N}(u) = 0$, $(t, x) \in \mathbf{R}_+^1 \times \mathbf{R}^d$, $d \geq 2$, where \mathcal{L} is assumed to be a constant coefficient operator, whose symbol depends only on the radial Fourier variable $|\xi|$. Such waves have been constructed in various situations, but for the sake of the argument, we will restrict our attention to the case of the beam equation⁵, (18). In fact, some of these constructions were carried over for traveling waves, which are not necessarily homoclinic to zero⁶, but we will sidestep this issue, with the hope that the situations can be easily reduced to the homoclinic to zero waves scenario.

⁴except at the natural point $|\omega| = \omega_{p,d}^{beam}$, where one has orbital instability, but linear stability

⁵where we might consider more general nonlinearities $\mathcal{N}(u)$, instead of the standard $|u|^{p-1}u$

⁶see for example [38], [33], which construct traveling waves for Swift-Hohenberg nonlinearity homoclinic to 1

More precisely, for the beam equation (18) solutions in the form $\varphi(x - \vec{c}t)$, which are radial in the $x' = (x_2, \dots, x_n)$ variable have been constructed by a variational argument (see [26]). More general nonlinearities have also been considered in the literature, for example by Smets - van den Berg, [38], Santra-Wei, [33] and Karageorgis-McKenna, [21]. It is easy to see that by the invariance under rotation, one may reduce matters to solutions in the form $\varphi(x_1 - ct, x_2, \dots, x_d)$. For the record, these radial in x' and even in x_1 solutions satisfy

$$\mathcal{L}\varphi + c^2\partial_1^2\varphi + \mathcal{N}(\varphi) = 0.$$

Linearizing around those solutions by $u = \varphi(x_1 - ct, x') + \mathbf{v}(x_1 - ct, x')$ yields the linearized problem

$$(19) \quad \mathbf{v}_{tt} - 2c\mathbf{v}_{tx_1} + (\mathcal{L} + c^2\partial_1^2 + \mathcal{N}'(\varphi))\mathbf{v} = 0$$

It is conceivable that under some general assumptions, the self-adjoint operator $\mathcal{H} = \mathcal{L} + c^2\partial_1^2 + \mathcal{N}'(\varphi)$ will have a single and simple negative eigenvalue, with an eigenfunction ϕ (which is radial in x' and even in x_1) corresponding to it, as well as d dimensional kernel, spanned by the vectors $\{\partial_1\varphi, \dots, \partial_d\varphi\}$. On the other hand, $J = -\partial_{x_1}$ will be a skew-symmetric operator.

This puts us in a favorable situation with respect to Theorem 1, as long as we can identify the spaces X^\pm , so that (12), (13), (14) hold true. Set

$$X^+ = \{f : \mathbf{R}^d \rightarrow \mathbf{R}^1 : f(x_1, x') = f(-x_1, x')\}, \quad X^- = \{f : \mathbf{R}^d \rightarrow \mathbf{R}^1 : f(x_1, x') = -f(-x_1, x')\}$$

and observe that $\mathcal{H} : X^\pm \rightarrow X^\pm$, $\psi_0 := \partial_1\varphi/\|\partial_1\varphi\|$ is odd in x_1 , while

$\phi, \psi_1 = \partial_2\varphi/\|\partial_2\varphi\|, \dots, \psi_{d-1} := \partial_d\varphi/\|\partial_d\varphi\|$ are even in x_1 . In addition, (15) holds as well, since

$$\langle \psi_j, J\psi_0 \rangle = -\frac{c}{\|\partial_1\varphi\|\|\partial_j\varphi\|} \langle \partial_j\varphi, \partial_1^2\varphi \rangle = \frac{c}{\|\partial_1\varphi\|\|\partial_j\varphi\|} \langle \partial_j\partial_1\varphi, \partial_1\varphi \rangle = 0, \quad j = 2, \dots, d.$$

Thus, Theorem 1 applies and in fact, it yields

$$\langle \mathcal{H}^{-1}J\psi_0, J\psi_0 \rangle = -\frac{1}{2c\|\partial_1\varphi\|^2} \langle \partial_c\varphi, \partial_1^2\varphi \rangle = \frac{1}{2c\|\partial_1\varphi\|^2} \partial_c[\|\partial_1\varphi\|^2] = \frac{\partial_c\|\partial_1\varphi\|}{c\|\partial_1\varphi\|}.$$

Hence, according to Theorem 1 instability occurs whenever $c^{-1}\partial_c[\|\partial_1\varphi\|] \geq 0$ or else, if

$$|c| < \frac{1}{2\sqrt{-\frac{\partial_c\|\partial_1\varphi\|}{c\|\partial_1\varphi\|}}}.$$

2.6. Comments and open problems. We would like to point out a few open problems, which should be tractable through the methods developed here. First, while Theorem 4 gives a complete answer for the stability of standing waves (i.e. $c = 0$) of the KGZ system, it does not address the same question for the general case of standing-traveling waves $(\varphi_{\omega,c}, \psi_{\omega,c})$. Note that the traveling waves case (i.e. $\omega = 0$) was considered in our earlier paper, [39]. The difficulty in applying Theorem 1 stems from the complicated structure⁷ of the operator J , which does not satisfy the requirements needed to perform the spectral analysis.

Another problem, in the whole space context is to study the stability of excited standing waves, which do not necessarily have the good spectral properties of ground states - for example L_+ may have more than one negative eigenvalue. As we showed in Theorem 3 above, it is possible to draw some conclusions in some special situations (equivariant solutions, perturbed by equivariant functions), but in general this remains a hard question to address.

Another line of research, for which Theorem 1 has proved itself useful, is the issue for linear stability of spatially periodic standing waves for the same models. Note that in collaboration with Hakkaev, [13] we have used the companion results in [39], to characterize the stable traveling

⁷and for example does not allow the splitting $L^2 = X^+ \oplus X^-$.

wave solutions of the Boussinesq equation and the Klein-Gordon-Zakharov system. This paper is currently under submission.

The paper is organized as follows. In Section 3, we prove the instability claims of Theorem 1. In Section 4, we show that the stability occurs in the complementary set. The last two sections contain the proofs of the applications to the Klein-Gordon equation and the Klein-Gordon-Zakharov system.

Postscriptum: After this paper was finished, Hakkaev, [14] has used the results obtained herein to completely characterize the stability of standing waves for the quadratic Klein-Gordon equation.

We should mention here the paper [2], which has been posted some time after we have prepared the final version of this paper. In it, the authors generalize some of the conclusions of Theorem 1, most notably in the case when \mathcal{H} has more than one negative eigenvalue. It should be noted however that these results are achieved under the assumption of an absence of an essential spectrum, which essentially reduces the applicability to the cases of spatially periodic waves. In addition, the case of the equality in alternative three in Theorem 1 (which is a stable configuration according to it) cannot be handled, due to a technical assumption in [2]. The interested reader should consult [2] for further details.

3. PROOF OF THEOREM 1: THE CASES OF INSTABILITY

Since we assume that \mathcal{H} has a simple and single negative eigenvalue, the operator \mathcal{H} will have a co-dimension one subspace on which it is non-negative. Denote the corresponding orthogonal projection by $P_{\geq 0} := \chi_{(0, \infty)}(\mathcal{H})[L^2]$, which can be written as $P_{\geq 0}f = f - \langle f, \phi \rangle \phi$.

Thus, we will be looking for a solution to (16) in the form $\lambda \neq 0$ real and $\psi = \phi + v$, where $v = P_{\geq 0}v \in P_{\geq 0}[L^2]$ is the projection onto the stable subspace and ϕ is the unstable eigenvector. Plugging this in (16) yields

$$(20) \quad (\lambda^2 + 2z\lambda J + \mathcal{H})v + (\lambda^2 - \delta^2)\phi + 2z\lambda J\phi = 0$$

Since by our spectral assumptions we can decompose $L^2 = \{\phi\} \oplus P_{\geq 0}(L^2)$, it follows that (20) holds if it holds along ϕ and after applying the projection $P_{\geq 0}$. Thus, the equation (20) is satisfied, if and only if the following pair of equations are both satisfied

$$(21) \quad \langle v, J\phi \rangle = \frac{\lambda^2 - \delta^2}{2z\lambda}$$

$$(22) \quad (\lambda^2 + 2z\lambda P_{\geq 0}JP_{\geq 0} + P_{\geq 0}\mathcal{H}P_{\geq 0})v = -2z\lambda P_{\geq 0}[J\phi].$$

Here, we have used the fact that $\langle \phi, J\phi \rangle = 0$, which was established before. Introduce the notation $\mathcal{H}_{\geq 0} := P_{\geq 0}\mathcal{H}P_{\geq 0}$, $\mathcal{H}_{> 0} := P_{> 0}\mathcal{H}P_{> 0}$, etc.

In the equation (22), we may in fact solve for v , since the operator $\mathcal{H}_{\geq 0} + \lambda^2 + 2z\lambda J_{\geq 0} : P_{\geq 0}[L^2] \rightarrow P_{\geq 0}[L^2]$ is invertible. This follows from the following proposition, which appears as Theorem in [3].

Proposition 1. *Assume that A is a closed, densely defined (not necessarily self-adjoint) operator on a Hilbert space, which is bounded from below ($\inf_{u \in D(A): \|u\|=1} \langle Au, u \rangle > -\infty$). Define its self-adjoint part $H = \Re A = \frac{1}{2}[A + A^*]$. Then*

$$\inf\{\Re \lambda : \lambda \in \sigma(A)\} \geq \inf \sigma(H).$$

In particular, if $H > 0$, then A is invertible.

Applying Proposition 1 to $A = \mathcal{H}_{\geq 0} + \lambda^2 + 2z\lambda J_{\geq 0}$ (note that its self-adjoint part $H = H_{\geq 0} + \lambda^2 > 0$) implies the invertibility and hence $v = -2z\lambda(\mathcal{H}_{\geq 0} + \lambda^2 + 2z\lambda J_{\geq 0})^{-1}[P_{\geq 0}J\phi]$. Thus, as a consequence of (21) and (22), we will have a solution of (16), if the following function

$$(23) \quad \mathcal{G}(\lambda) = \langle (\mathcal{H}_{\geq 0} + \lambda^2 + 2z\lambda J_{\geq 0})^{-1}[P_{\geq 0}J\phi], P_{\geq 0}J\phi \rangle + \frac{\lambda^2 - \delta^2}{4z^2\lambda^2},$$

has a root in $(0, \infty)$. Indeed, suppose $\mathcal{G}(\lambda_0) = 0$, for some $\lambda_0 > 0$. Take then $v_0 = -2z\lambda_0(\mathcal{H}_{\geq 0} + \lambda_0^2 + 2z\lambda_0 J_{\geq 0})^{-1}[P_{\geq 0}(J\phi)] \in P_{\geq 0}[L^2]$. This clearly satisfies (22) with $\lambda = \lambda_0$. Next,

$$\begin{aligned} 2z\lambda_0 \langle v_0, J\phi \rangle &= 2z\lambda_0 \langle v_0, P_{\geq 0}(J\phi) \rangle = \\ &= -4z^2\lambda_0^2 \langle (\mathcal{H}_{\geq 0} + \lambda_0^2 + 2z\lambda_0 J_{\geq 0})^{-1}[P_{\geq 0}(J\phi)], P_{\geq 0}(J\phi) \rangle = \lambda_0^2 - \delta^2, \end{aligned}$$

where in the last identity, we have used $\mathcal{G}(\lambda_0) = 0$. Thus,

$$\langle v_0, J\phi \rangle = \frac{\lambda_0^2 - \delta^2}{2z\lambda_0}$$

and hence (21) is satisfied as well. Thus, we have shown that (16) holds with $\psi = \phi + v_0$, $\lambda = \lambda_0$, provided that the function \mathcal{G} vanishes somewhere in $(0, \infty)$.

Our next lemma states the intuitively clear statement that \mathcal{G} is jointly continuous on $\mathbf{R}_+^1 \times \mathbf{R}_+^1$.

Lemma 1. *Consistent with the discussion above, define the function*

$$\mathcal{G}(z; \lambda) = \langle (\mathcal{H}_{\geq 0} + \lambda^2 + 2z\lambda J_{\geq 0})^{-1}[J\phi], J\phi \rangle + \frac{\lambda^2 - \delta^2}{4z^2\lambda^2},$$

For any sequence $(z_n, \lambda_n) \rightarrow (z_0, \lambda_0)$, so that $z_n, z_0, \lambda_n, \lambda_0 \in \mathbf{R}_+^1$, we have

$$\lim_n \mathcal{G}(z_n; \lambda_n) = \mathcal{G}(z_0, \lambda_0).$$

The statement and the proof are identical to Lemma 1, [39], so we omit it.

We will prove our instability claims, by showing that the function \mathcal{G} changes sign in $(0, \infty)$, under the conditions specified in Theorem 1. This fact, together with the continuity of \mathcal{G} will guarantee the existence of a root and hence the proof of Theorem 1 will be complete. Thus, we will consider the behavior of \mathcal{G} at $\lambda = \infty$ and $\lambda = 0$.

3.1. Behavior of $\mathcal{G}(\lambda)$ at $\lambda = \infty$. Clearly $\lim_{\lambda \rightarrow \infty} \frac{\lambda^2 - \delta^2}{4z^2\lambda^2} = \frac{1}{4z^2} > 0$. Next, we will show that $\lim_{\lambda \rightarrow \infty} \langle (\mathcal{H}_{\geq 0} + \lambda^2 + 2z\lambda J_{\geq 0})^{-1}[P_{\geq 0}J\phi], P_{\geq 0}J\phi \rangle = 0$, which will imply that

$$\lim_{\lambda \rightarrow \infty} \mathcal{G}(\lambda) = \frac{1}{4z^2} > 0.$$

We need the following proposition, which is very similar to Proposition 3 of [39].

Proposition 2. *For every $\lambda > 0$, we have $H_{\geq 0} + \lambda^2 + 2z\lambda J_{\geq 0} : P_{\geq 0}(L^2) \rightarrow P_{\geq 0}(L^2)$. In this co-dimension one subspace, the operator has an inverse, which obeys the estimate*

$$(24) \quad \|(H_{\geq 0} + \lambda^2 + 2z\lambda J_{\geq 0})^{-1}\|_{P_{\geq 0}(L^2) \rightarrow P_{\geq 0}(L^2)} \leq \lambda^{-2}$$

Proof. We have already checked the invertibility. Let $g \in P_{\geq 0}(L^2)$ be an arbitrary real-valued function and $f = (H_{\geq 0} + \lambda^2 + 2z\lambda J_{\geq 0})^{-1}[g] \in P_{\geq 0}(L^2)$, so that

$$(H_{\geq 0} + \lambda^2 + 2z\lambda J_{\geq 0})f = g.$$

Note f is real valued as well. Taking dot product with f yields (noting $\langle J_{\geq 0}f, f \rangle = \langle Jf, f \rangle = 0$)

$$\lambda^2 \|f\|^2 \leq \langle (H_{\geq 0} + \lambda^2)f, f \rangle = \langle (H_{\geq 0} + \lambda^2 + 2z\lambda J_{\geq 0})f, f \rangle = \langle f, g \rangle \leq \|f\| \|g\|,$$

whence $\|(H_{\geq 0} + \lambda^2 + 2z\lambda J_{\geq 0})^{-1}[g]\| = \|f\| \leq \lambda^{-2} \|g\|$, as stated. \square

Based on Proposition 2, we clearly see that

$$\limsup_{\lambda \rightarrow \infty} |\langle (\mathcal{H}_{\geq 0} + \lambda^2 + 2z\lambda J_{\geq 0})^{-1} [P_{\geq 0} J\phi], P_{\geq 0} J\phi \rangle| \leq \limsup_{\lambda \rightarrow \infty} \lambda^{-2} \|P_{\geq 0} J\phi\|^2 = 0,$$

which completes the analysis near $\lambda = \infty$.

3.2. Behavior of \mathcal{G} close to $\lambda = 0$. The behavior close to $\lambda = 0$ is a more complicated matter. Let us set $\lambda = \varepsilon$, where $0 < \varepsilon \ll 1$. Clearly

$$\frac{\varepsilon^2 - \delta^2}{4z^2\varepsilon^2} = -\frac{\delta^2}{4z^2\varepsilon^2} + O(1)$$

Now, we would like to study the Laurent expansion at zero for the function $\varepsilon \rightarrow \langle (\mathcal{H}_{\geq 0} + \varepsilon^2 + 2z\varepsilon J_{\geq 0})^{-1} [P_{\geq 0} J\phi], P_{\geq 0} J\phi \rangle$. Let $z \in P_{\geq 0}(L^2)$ be such that

$$(\mathcal{H}_{\geq 0} + \varepsilon^2 + 2z\varepsilon J_{\geq 0})z = P_{\geq 0}[J\phi]$$

Represent $z = a_0\psi_0 + a_1\psi_1 + \dots + a_n\psi_n + q$, where a_0, \dots, a_n are scalars and $q \in P_{>0}(L^2)$. In particular $q \perp \text{span}\{\psi_0, \dots, \psi_n, \phi\}$. We have

$$(25) \quad \varepsilon^2(a_0\psi_0 + \dots + a_n\psi_n) + 2z\varepsilon(a_0J_{\geq 0}\psi_0 + \dots + a_nJ_{\geq 0}\psi_n) + (\mathcal{H}_{\geq 0} + \varepsilon^2 + 2z\varepsilon J_{\geq 0})q = P_{\geq 0}[J\phi].$$

Take a dot product with ψ_0, \dots, ψ_n in (25). Clearly $\langle J\psi_j, \psi_k \rangle = 0 = \langle J\phi, \psi_k \rangle$, whenever $j, k \in [1, n]$, because $J\psi_j, J\phi \in X^+$, $\psi_k \in X^-$, $X^+ \perp X^-$. Finally, $\langle J\psi_0, \psi_k \rangle = 0$ by assumption. We get the equations

$$(26) \quad a_0\varepsilon^2 - 2z\varepsilon \langle q, J\psi_0 \rangle = \langle J\phi, \psi_0 \rangle$$

$$(27) \quad a_k\varepsilon^2 - 2z\varepsilon \langle q, J\psi_k \rangle = 0; \quad k = 1, \dots, n,$$

while applying the projection $P_{>0}$ yields the equation for q

$$(28) \quad (\mathcal{H}_{>0} + \varepsilon^2 + 2z\varepsilon J_{>0})q = P_{>0}[J\phi] - 2z\varepsilon P_{>0}[a_0J\psi_0 + \dots + a_nJ\psi_n].$$

Now, we have required in (14) that $H_{>0} \geq \kappa^2 > 0$, which implies that $\mathcal{H}_{>0}^{-1} \leq \kappa^{-2} = O(1)$ in terms of ε . Thus,

$$(\mathcal{H}_{>0} + \varepsilon^2 + 2z\varepsilon J_{>0})^{-1} = \mathcal{H}_{>0}^{-1} + O(\varepsilon),$$

in the sense of operator norms. Thus, we may write from (28)

$$(29) \quad q = -2z\varepsilon \sum_{k=0}^n a_k \mathcal{H}_{>0}^{-1}[J\psi_k] + O(\varepsilon^2 \sum |a_k|) + O(1).$$

From (27) however, we have for $l \geq 1$,

$$(30) \quad a_l = \frac{2z}{\varepsilon} \langle q, J\psi_l \rangle = -4z^2 \sum_{k=0}^n a_k \langle \mathcal{H}_{>0}^{-1}[J\psi_k], J\psi_l \rangle + O(\varepsilon \sum |a_k|) + O(\varepsilon^{-1}).$$

Note however that $\langle \mathcal{H}_{>0}^{-1}[J\psi_0], J\psi_l \rangle = 0$, since $J\psi_0 \in X^+$, $J\psi_l \in X^-$ and hence $\mathcal{H}_{>0}^{-1}[J\psi_0] \perp J\psi_l$. Thus, the first sum above runs only on $k \geq 1$. Thus, we get a linear system for a_1, \dots, a_n , of the form

$$B\vec{a} = O(\varepsilon \sum_{k=1}^n |a_k|) + O(\varepsilon^{-1})$$

where

$$B = Id + 4z^2 \{ \langle \mathcal{H}_{>0}^{-1}[J\psi_k], J\psi_l \rangle \}_{kl} \geq Id,$$

by the positivity of the matrix $\{ \langle \mathcal{H}_{>0}^{-1}[J\psi_k], J\psi_l \rangle \}_{kl}$. Thus,

$$a_k = O(\varepsilon |a_0|) + O(\varepsilon^{-1}), \quad k \in [1, n].$$

Going back to (29) and based on the last identity, we have that

$$(31) \quad q = -2z\varepsilon a_0 \langle \mathcal{H}_{>0}^{-1}[J\psi_0], J\psi_0 \rangle + O(\varepsilon^2 a_0) + O(1).$$

Now, from (26) and (31), we conclude that

$$a_0 \varepsilon^2 + 4z^2 \varepsilon^2 a_0 \langle \mathcal{H}_{>0}^{-1}[J\psi_0], J\psi_0 \rangle = \langle J\phi, \psi_0 \rangle + O(\varepsilon) + O(a_0 \varepsilon^3),$$

which implies

$$(32) \quad a_0 = \frac{1}{\varepsilon^2} \frac{\langle J\phi, \psi_0 \rangle}{1 + 4z^2 \langle \mathcal{H}_{>0}^{-1}[J\psi_0], J\psi_0 \rangle} + O(\varepsilon^{-1})$$

In particular, from (31), $q = O(\varepsilon^{-1})$. Clearly then the function of interest becomes

$$\begin{aligned} & \langle (\mathcal{H}_{\geq 0} + \varepsilon^2 + 2z\varepsilon J_{\geq 0})^{-1}[P_{\geq 0}J\phi], P_{\geq 0}J\phi \rangle = a_0 \langle \psi_0, P_{\geq 0}J\phi \rangle + O(\varepsilon^{-1}) = \\ & = \frac{1}{\varepsilon^2} \frac{\langle J\phi, \psi_0 \rangle^2}{1 + 4z^2 \langle \mathcal{H}_{>0}^{-1}[J\psi_0], J\psi_0 \rangle} + O(\varepsilon^{-1}) \end{aligned}$$

Thus,

$$\mathcal{G}(\varepsilon) = \frac{1}{\varepsilon^2} \left(\frac{\langle \phi, J\psi_0 \rangle^2}{1 + 4z^2 \langle \mathcal{H}_{>0}^{-1}[J\psi_0], J\psi_0 \rangle} - \frac{\delta^2}{4z^2} \right) + O(\varepsilon^{-1})$$

Thus, it is clear that if

$$(33) \quad \frac{\langle \phi, J\psi_0 \rangle^2}{1 + 4z^2 \langle \mathcal{H}_{>0}^{-1}[J\psi_0], J\psi_0 \rangle} - \frac{\delta^2}{4z^2} < 0,$$

we have $\limsup_{\varepsilon \rightarrow 0^+} \mathcal{G}(\varepsilon) = -\infty$ and hence \mathcal{G} has a root in $(0, \infty)$. In order to analyze (33), we write it in the equivalent form

$$4z^2 \left(\frac{\langle \phi, J\psi_0 \rangle^2}{\delta^2} - \langle \mathcal{H}_{>0}^{-1}[J\psi_0], J\psi_0 \rangle \right) < 1.$$

Note that

$$\frac{\langle \phi, J\psi_0 \rangle^2}{\delta^2} - \langle \mathcal{H}_{>0}^{-1}[J\psi_0], J\psi_0 \rangle = -\langle \mathcal{H}^{-1}[J\psi_0], J\psi_0 \rangle$$

Thus, (33) is equivalent to

$$-4z^2 \langle \mathcal{H}^{-1}J\psi_0, J\psi_0 \rangle \leq 1.$$

The last inequality is satisfied for all z , if $\langle \mathcal{H}^{-1}[J\psi_0], J\psi_0 \rangle \geq 0$. It is also satisfied if $\langle \mathcal{H}^{-1}[J\psi_0], J\psi_0 \rangle < 0$, if $|z| < \frac{1}{2\sqrt{-\langle \mathcal{H}^{-1}[J\psi_0], J\psi_0 \rangle}}$. This was exactly the claim in Theorem 1.

4. PROOF OF THEOREM 1: THE CASES OF STABILITY

In this section, we prove the stability of the standing waves, under the conditions of Theorem 1. In fact, our proof follows almost step by step the proof of the corresponding result for traveling waves, that we have obtained earlier in [39]. Of course, there are some technical details in the standing wave case that are different and we address them herein.

Since Theorem 1 addressed the case of $\langle \mathcal{H}^{-1}[J\psi_0], J\psi_0 \rangle \geq 0$, assume that $\langle \mathcal{H}^{-1}[J\psi_0], J\psi_0 \rangle < 0$. Furthermore, the case $|z| < z^*(\mathcal{H})$, where $z^*(\mathcal{H}) = \frac{1}{2\sqrt{-\langle \mathcal{H}^{-1}[J\psi_0], J\psi_0 \rangle}}$ was shown to be unstable

as well. Define the sets of reals

$$\begin{aligned}\mathcal{A}^{unstable} &:= \{z : (z, J, \mathcal{H}) \text{ is unstable}\} \\ \mathcal{A}^{stable} &:= \{z : (z, J, \mathcal{H}) \text{ is stable}\}\end{aligned}$$

We have shown in Section 3 that $(-z^*(\mathcal{H}), z^*(\mathcal{H})) \subset \mathcal{A}^{unstable}$. Thus, the stability claims in Theorem 1 is that $\mathcal{A}^{stable} = \{z : |z| \geq z^*(\mathcal{H})\}$ and this is what we will prove in this Section.

First, we need some results from the Shkalikov's theory for stability of quadratic pencils, presented in [37], see also the exposition in [39].

4.1. Shkalikov's theory for stability of quadratic pencils. We follow almost verbatim the presentation of Shkalikov's theory in our earlier paper, [39]. We are especially interested in his index formula, which relates the number of unstable eigenvalues of \mathcal{H} with the number of unstable eigenvalues of the pencil defined in (16).

Shkalikov, [37] introduced a (more general) quadratic operator pencil in the form

$$A(\lambda) = \lambda^2 F + (D + iG)\lambda + T,$$

where the coefficients F, D, G, T are operators on a Hilbert space H , satisfying the following conditions

- (i) F - bounded, invertible and self-adjoint
- (ii) $(T, \text{Dom}(T))$ - is self-adjoint and invertible
- (iii) $D \geq 0, G$ are symmetric; $\text{Dom}(D), \text{Dom}(G) \subset \text{Dom}(T)$ and D, G are T -bounded operators,

where we have used the notion of T -bounded operator, which is the following

Definition 2. We say that an operator M is T -bounded, if

$$|T|^{-1/2} M |T|^{-1/2} \in \mathcal{B}(H)$$

We say that ξ is in the resolvent set, i.e. $\xi \in \rho(A)$, if $A(\xi)$, with domain $\text{Dom}(T)$ is invertible.

Note that in the cases of interest to us $D = 0$!

Introduce the associated quadratic pencil

$$\hat{A}(\lambda) := \lambda^2 \hat{F} + \lambda(\hat{D} + i\hat{G}) + I$$

where $\hat{F} = |T|^{-1/2} F |T|^{-1/2}$, $\hat{G} = |T|^{-1/2} G |T|^{-1/2}$. We will be also interested in the spectrum with respect to a smoother space. Namely, introduce the space H_{-1} with a norm

$$\|x\|_{H_{-1}} := \||T|^{-1/2} x\|_H$$

It is shown in [37] that the spectrum of A in H_{-1} coincides with the spectrum of \hat{A} considered on the space H . Note that in Definition 1, we only consider smooth enough solutions ψ anyway. Thus, we need to count the number of "unstable" eigenvalues of \hat{A} .

The following result is an immediate corollary of Theorem 3.7 in [37].

Theorem 6. (Theorem 3.7, [37]) Suppose the coefficients of the pencil A , F, D, G, T satisfy conditions (i) – (iii) above. Suppose that the numbers of negative eigenvalues of F and T , $\nu(F)$ and $\nu(T)$ respectively are finite.

Then, the spectrum of $A(\lambda)$ in the open right-half plane $\mathcal{C}_r = \{z : \Re z > 0\}$, considered upon the space H_{-1} , consists of eigenvalues only. Moreover, the total algebraic multiplicity of all eigenvalues lying in \mathcal{C}_r satisfies

$$k(\hat{A}) \leq \nu(T) + \nu(F).$$

In our approach, we will be interested in the case $F = Id$, $D = 0$, $G = -2ziJ$, $T = \mathcal{H}$. Note that they all satisfy conditions (i) – (iii), except that the operator $T = \mathcal{H}$ is not invertible. This case is also covered by Shkalikov, see Theorem 4.2,[37] under our assumption (12). We provide a brief sketch of the proof, based on Theorem 6.

Indeed, since $Ker(\mathcal{H}) \neq \{0\}$, one needs to consider $\mathcal{H}_\tau := \mathcal{H} + \tau Id$ for $0 < \tau \ll 1$, so that $Ker(\mathcal{H}_\tau) = \{0\}$. In order to apply Theorem 6, we need to check that $F = Id, G = -2izJ$ are \mathcal{H}_τ bounded. This amounts to showing that $|\mathcal{H}_\tau|^{-1/2}J|\mathcal{H}_\tau|^{-1/2} \in \mathcal{B}(L^2)$. But J is a bounded operator by assumption, hence \mathcal{H}_τ bounded as well.

We can now apply Theorem 6 to \hat{A}_τ to conclude

$$k(\hat{A}_\tau) \leq \nu(\mathcal{H}_\tau) + \nu(Id) = 1,$$

for all small enough $\tau > 0$. Since the eigenvalues depend continuously on τ (see [22], Chapter 7), we take a limit as $\tau \rightarrow 0+$ to get the desired inequality $k(\hat{A}) \leq 1$.

To recapitulate, we have shown that for fixed real z and under (12), the equation

$$(34) \quad \lambda^2\psi + 2\lambda zJ\psi + \mathcal{H}\psi = 0, \quad (\lambda, \psi) \in \mathbf{R}_+^1 \times L^2,$$

has at most one solution with $\Re\lambda > 0$.

Finally, we show that in this case, any solution λ must be real. Indeed, suppose $\lambda \neq \bar{\lambda}$ is an eigenvalue for the pencil. That is, there is $\psi \in D(H) : \lambda^2\psi + 2\lambda zJ\psi + \mathcal{H}\psi = 0$. Taking a complex conjugate and in view of $\overline{\mathcal{H}\psi} = \mathcal{H}\bar{\psi}$, $\overline{J\psi} = J\bar{\psi}$, we see that

$$\bar{\lambda}^2\bar{\psi} + 2\bar{\lambda}zJ\bar{\psi} + H\bar{\psi} = 0$$

Thus, $(\bar{\lambda}, \bar{\psi})$ is another solution to (34), with $\Re\bar{\lambda} > 0$ and hence $k(A) \geq 2$, in contradiction with the inequality $k(A) \leq 1$. Thus, λ must be real and we have established

Corollary 2. *For $z \in \mathbf{R}^1$, the equation $\lambda^2\psi + 2z\lambda J\psi + \mathcal{H}\psi = 0$, $\lambda \in \mathcal{C}$, $\psi \in D(H)$ has at most one solution (λ, ψ) with $\Re\lambda > 0$. Moreover, such a pair will have λ real, $\lambda > 0$.*

Thus, we have proved that for instabilities to occur, we need to have a solution (λ, ψ) of the problem (16), so that $\lambda > 0, \psi \in D(\mathcal{H})$.

4.2. The positive zeros of \mathcal{G} are exactly the unstable eigenvalues of the pencil. The analysis of this section is pretty similar to the one performed in the previous section. Of course, in Theorem 1 we were only looking for instabilities, whereas now we need to show that if $\mathcal{G}(\lambda_0) = 0$ if and only if λ_0 is an unstable eigenvalue for the pencil.

Proposition 3. *The positive number $\lambda_0 > 0$ is an unstable eigenvalue for (z, J, \mathcal{H}) if and only if the function*

$$\mathcal{G}(z; \lambda) = \langle (\mathcal{H}_{\geq 0} + \lambda^2 + 2z\lambda J_{\geq 0})^{-1}[J\phi], J\phi \rangle + \frac{\lambda^2 - \delta^2}{4z^2\lambda^2},$$

vanishes at λ_0 . In view of Corollary 2, the pencil (z, J, \mathcal{H}) is unstable if and only if $\mathcal{G}(z, \lambda) = 0$ for some $\lambda > 0$.

Proof. The first observation is that if (λ, ψ) solves

$$(35) \quad (\mathcal{H} + \lambda_0^2 + 2z\lambda_0 J)\psi = 0,$$

then $\langle \psi, \phi \rangle \neq 0$. Assume otherwise, that is $\psi \perp \phi$. It follows that

$$(\mathcal{H}_{\geq 0} + \lambda_0^2 + 2z\lambda_0 J_{\geq 0})\psi = 0$$

and $\mathcal{H}_{\geq 0} + \lambda_0^2 + 2z\lambda_0 J_{\geq 0} : \{\phi\}^\perp \rightarrow \{\phi\}^\perp$. Apply Proposition 2 to this operator. Since $J_{\geq 0}^* = -J_{\geq 0}$, it follows that the self-adjoint part is $\mathcal{H}_{\geq 0} + \lambda_0^2 \geq \lambda_0^2 Id$ and hence invertible. Thus, $\mathcal{H}_{\geq 0} + \lambda_0^2 + 2z\lambda_0 J_{\geq 0}$ is invertible on $\{\phi\}^\perp$ and hence $\psi = 0$, a contradiction.

Since solutions of (35) are up to a multiplicative constant, we can write $\psi = \phi + v, v \perp \phi, v \in P_{\geq 0}(L^2)$ and hence

$$(36) \quad (\mathcal{H} + \lambda_0^2 + 2z\lambda_0 J)v = (\delta^2 - \lambda_0^2)\phi - 2z\lambda_0 J\phi.$$

Based on that, we derive the pair of equations (21), (22). The difference is that now, we have started with a solution (λ_0, ψ) of (35) and concluded (21) and (22). As in the proof of Theorem 1, we derive from (22) that $v = -2z(\mathcal{H}_{\geq 0} + \lambda_0^2 + 2z\lambda_0 J_{\geq 0})^{-1}[J\phi]$, which then in view of (21) implies $\mathcal{G}(z; \lambda_0) = 0$. The reverse implications is clear as well and so, Proposition 3 is established. \square

4.3. Proof of $(-\infty, -z^*(\mathcal{H})) \cup (z^*(\mathcal{H}), \infty) \subset \mathcal{A}^{stable}$. This is really similar to Section 3.3 in [39]. We have already established that for fixed $z : |z| > z^*(\mathcal{H})$,

$$\lim_{\lambda \rightarrow \infty} \mathcal{G}(z, \lambda) = \frac{1}{4z^2}; \quad \lim_{\lambda \rightarrow 0^+} \mathcal{G}(z, \lambda) = +\infty$$

By the continuity of the function $\lambda \rightarrow \mathcal{G}(z, \lambda)$, we have three alternatives

- (1) $\mathcal{G}(z, \lambda) > 0$ for all $\lambda \in (0, \infty)$.
- (2) There are $0 < \lambda_1 < \lambda_2 < \infty$, so that $\mathcal{G}(z, \lambda_1) = 0 = \mathcal{G}(z, \lambda_2)$
- (3) $\mathcal{G}(z, \lambda) \geq 0$, but $\mathcal{G}(z, \lambda_0) = 0$ for some $\lambda_0 > 0$, i.e. λ_0 is a double root for \mathcal{G} .

The alternative (1) above is what we need to prove, so we proceed to refute options (2) and (3). In fact, option (2) is easily seen to be impossible due to Proposition 3 and the Shkalikov's theory. Indeed, assuming two separate zeros of \mathcal{G} implies that (34) will have different solutions (λ_1, ψ_1) , (λ_2, ψ_2) , in contradiction with the Corollary 2 of the Shkalikov's theory.

Now, it remains to show that option (3) is impossible as well. Indeed, assuming that $\lambda_0 > 0$ is a double root, construct $\mathcal{H}_\mu := \mathcal{H} - \mu^2 \langle \cdot, \phi \rangle \phi$. Clearly, the operator $\mathcal{H}_\mu, 0 < \mu \ll 1$ satisfies all the assumptions in our theory and therefore, it should be that the corresponding function $\mathcal{G}_{\mathcal{H}_\mu}$ will have no more than one root in $(0, \infty)$. However since $P_{\geq 0}\mathcal{H}_\mu P_{\geq 0} = P_{\geq 0}\mathcal{H}P_{\geq 0} = \mathcal{H}_{\geq 0}$

$$G_{\mathcal{H}_\mu}(z, \lambda) = \langle (\mathcal{H}_{\geq 0} + \lambda^2 + 2z\lambda J_{\geq 0})^{-1}[J\phi], J\phi \rangle + \frac{\lambda^2 - \delta^2 - \mu^2}{4z^2\lambda^2} = G(z, \lambda) - \frac{\mu^2}{4z^2\lambda^2}.$$

Moreover, since $z > z^*(\mathcal{H})$, it follows that $z > z^*(\mathcal{H}_\mu)$ for all $\mu \ll 1$ (by the obvious continuity of $z^*(\mathcal{H}_\mu)$ with respect to μ). Thus, we still have $\lim_{\lambda \rightarrow 0^+} G_{\mathcal{H}_\mu}(z, \lambda) = +\infty$ for all $\mu \ll 1$.

Now, if $\mathcal{G}(z, \lambda_0) = 0$, clearly $G_{\mathcal{H}_\mu}(z, \lambda_0) < 0$ for all $\mu \ll 1$, implying the existence of at least two zeros $\lambda_1(\mu) < \lambda < \lambda_2(\mu)$ for $G_{\mathcal{H}_\mu}$, a contradiction with Corollary 2 applied to \mathcal{H}_μ . This leaves only the first alternative, whence we have a complete proof of our claim

$$(-\infty, -z^*(\mathcal{H})) \cup (z^*(\mathcal{H}), \infty) \subset \mathcal{A}^{stable}.$$

4.4. Proof of $z^*(\mathcal{H}) \in \mathcal{A}^{stable}$. This is an intuitively clear statement, if one believes in the continuity of the pair of eigenvalues $\pm\lambda(z)$ as a function of z . Indeed, the following scenario takes place: for $z \in (0, z^*(\mathcal{H}))$ corresponds to $\lambda(z)$ - real and as z approaches $z^*(\mathcal{H})$, the corresponding $\lambda(z)$ turns into zero as $z = z^*(\mathcal{H})$ and then $\lambda(z)$ turns into purely imaginary for $z > z^*(\mathcal{H})$. Thus, we expect $\lambda(z^*(\mathcal{H})) = 0$.

Regarding the formal proof, we proceed similar to [39]. We start with

Proposition 4. *For any $z > 0$, the function $\varepsilon \rightarrow \varepsilon^2 \mathcal{G}(z, \varepsilon)$ is real analytic in a neighborhood of zero. In fact \mathcal{G} has the Laurent expansion*

$$(37) \quad \mathcal{G}(z, \varepsilon) = \varepsilon^{-2} D_{-2}(z) + \sum_{j=-1}^{\infty} D_j(z) \varepsilon^j,$$

where

$$(38) \quad D_{-2}(z) = \frac{\langle \phi, J\psi_0 \rangle^2}{1 + 4z^2 \langle \mathcal{H}_{>0}^{-1}[J\psi_0], J\psi_0 \rangle} - \frac{\delta^2}{4z^2}.$$

In addition, the functions $\{D_j(z)\}$ are smooth functions of z and the radius of analyticity $r(z)$ can be chosen so that

$$\inf_{z \in [a,b] \subset (0,\infty)} r(z) \geq r_{a,b} > 0.$$

That is, whenever $[a, b] \subset (0, \infty)$, one has common nontrivial radius of analyticity for all functions $\{\mathcal{G}(z, \varepsilon)\}_{z \in [a,b]}$.

The proof of this proposition follows closely the similar Proposition 4 in [39], so we omit it. We would like to point out however, that the computation yielding the formula for $D_{-2}(z)$ was already needed in the course of the proof of the instability claims in Section 3, see for example the derivation of (32).

Denote for the rest of this section $z_0 = z^*(\mathcal{H}) > 0$. The first thing that one observes from (38) is that $D_{-2}(z_0) = 0$ and $D_{-2}(z) > 0$ for $z > z_0$, while $D_{-2}(z) < 0$ for $z < z_0$. Thus, consider the expansion

$$\mathcal{G}(z_0, \varepsilon) = D_{-2}(z_0)\varepsilon^{-2} + \sum_{j=-1}^{\infty} D_j(z_0)\varepsilon^j = \sum_{j=-1}^{\infty} D_j(z_0)\varepsilon^j$$

By the analyticity and since $\mathcal{G}(z_0, \varepsilon)$ is non-trivial, there must be a $k \in [-1, \infty)$, so that $D_k(z_0) \neq 0$. Let

$$k_0 = \min\{k \geq -1 : D_k(z_0) \neq 0\}.$$

We will show that $D_{k_0}(z_0) > 0$. To that end, take $z_j \rightarrow z_0 : z_j > z_0$. Since $z_j \in \mathcal{A}^{stable}$, we have that $\mathcal{G}(z_j, \varepsilon) > 0$ for all $\varepsilon > 0$. Letting $0 < \varepsilon < r(z_0)$, we have

$$0 \leq \limsup_j \mathcal{G}(z_j, \varepsilon) = \mathcal{G}(z_0, \varepsilon) = D_{k_0}(z_0)\varepsilon^{k_0} + O(\varepsilon^{k_0+1}).$$

This clearly implies that $D_{k_0}(z_0) > 0$, since otherwise we have $D_{k_0}(z_0) < 0$ and a limit as $\varepsilon \rightarrow 0+$ will yield a contradiction.

Finally, knowing that

$$\lim_{\lambda \rightarrow \infty} \mathcal{G}(z_0, \lambda) = +\infty, \quad \lim_{\lambda \rightarrow 0+} \lambda^{-k_0} \mathcal{G}(z_0, \lambda) = D_{k_0}(z_0) > 0,$$

implies that $z_0 \in \mathcal{A}^{stable}$. Indeed, otherwise there will be $\lambda_0 > 0$, so that $\mathcal{G}(z_0, \lambda_0) = 0$. By the fact that $\mathcal{G}(z, \cdot)$ is positive close to zero and at infinity (which was just established), it will follow that $\lambda \rightarrow \mathcal{G}(z_0, \lambda)$ has a double zero at λ_0 (by the impossibility for more than one zero, due to Proposition 3 and Corollary 2). But this leads to a contradiction, as we have argued in the previous section. Indeed, consider the function⁸ $\lambda \rightarrow \mathcal{G}_{\mathcal{H}_\mu}(z_0, \lambda)$ for all sufficiently small μ . We will have at least two zeros $0 < \lambda_1 < \lambda_0 < \lambda_2 < \infty$ and hence a contradiction with Corollary 2 of the Shkalikov's theory. The proof of Theorem 1 is complete.

⁸Note here that $z_0 = z^*(\mathcal{H}) \neq z^*(\mathcal{H}_\mu)$!

5. PROOF OF THEOREM 2, THEOREM 3 AND THEOREM 5

The proof of Theorem 2 consists of direct verification that the conditions of Theorem 1 are met for the linearized operator and then computing the appropriate index $\omega^*(\mathcal{H})$. First, note that, as was pointed out in Section 1.1.1, the relevant operator \mathcal{H} that we need to consider is in the form

$$\begin{aligned}\mathcal{H} &= \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}, J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ L_+ &= -\Delta + (1 - \omega^2) - p\varphi^{p-1} \\ L_- &= -\Delta + (1 - \omega^2) - \varphi^{p-1}\end{aligned}$$

Next, the operators L_{\pm} have the behavior postulated in Theorem 1, as it was shown by Weinstein, [41], Shatah, [35] and Kwong, [24]. Thus, it remains to compute the index $\omega^*(\mathcal{H})$.

Noting that $\psi_0 = \|\varphi_\omega\|^{-1} \begin{pmatrix} 0 \\ \varphi_\omega \end{pmatrix}$ we have

$$\langle \mathcal{H}^{-1} J \psi_0, J \psi_0 \rangle = \|\varphi_\omega\|^{-2} \langle L_+^{-1} \varphi_\omega, \varphi_\omega \rangle.$$

On the other hand, the defining equation for φ_ω is

$$(39) \quad -\Delta \varphi_\omega + (1 - \omega^2) \varphi_\omega - \varphi_\omega^p = 0.$$

Taking a derivative in ω yields immediately $L_+[\partial_\omega \varphi_\omega] - 2\omega \varphi_\omega = 0$, which implies that $L_+^{-1} \varphi_\omega = \frac{1}{2\omega} \partial_\omega \varphi_\omega$. Thus,

$$\langle \mathcal{H}^{-1} J \psi_0, J \psi_0 \rangle = \frac{1}{2\omega \|\varphi_\omega\|^2} \langle \partial_\omega \varphi_\omega, \varphi_\omega \rangle = \frac{1}{4\omega \|\varphi_\omega\|^2} \partial_\omega \|\varphi_\omega\|^2.$$

By scaling considerations and the uniqueness of the radial ground states, it follows that $\varphi_\omega(x) = (1 - \omega^2)^{\frac{1}{p-1}} \varphi_0(\sqrt{1 - \omega^2} x)$. Hence,

$$\|\varphi_\omega\|_{L^2}^2 = (1 - \omega^2)^{\frac{2}{p-1} - \frac{d}{2}} \|\varphi_0\|^2.$$

All in all,

$$\langle \mathcal{H}^{-1} J \psi_0, J \psi_0 \rangle = -\frac{\frac{1}{p-1} - \frac{d}{4}}{(1 - \omega^2)}$$

According to Theorem 1 then, we have instability, whenever $\frac{1}{p-1} - \frac{d}{4} \leq 0$, that is, when $p \geq 1 + \frac{4}{d}$. If on the other hand, $1 < p < 1 + \frac{4}{d}$, introduce

$$\omega_{p,d} = \sqrt{\frac{p-1}{4 - (p-1)(d-1)}} \in (0, 1)$$

as in the statement of Theorem 2. According to Theorem 1, we have stability exactly when

$$|\omega| \geq \frac{1}{2\sqrt{-\langle \mathcal{H}^{-1} J \psi_0, J \psi_0 \rangle}} = \frac{\sqrt{1 - \omega^2}}{\sqrt{\frac{4}{p-1} - d}}.$$

Solving this last inequality for ω yields exactly $|\omega| \geq \omega_{p,d}$, which was the claim. Theorem 2 is proved.

As far as Theorem 3 is concerned, the proof follows an identical argument as Theorem 2. Indeed, the only extra ingredient is the Mizumachi's result (see Proposition 3.1, [29]) that the operator \mathcal{H} satisfies the requirements of Theorem 1, namely the simplicity of its unique negative eigenvalue.

Regarding Theorem 5, we follow along the scheme of the proof of Theorem 2. Under the spectral assumptions made in Theorem 5, we can apply Theorem 1. We compute in the same fashion that

$$\langle \mathcal{H}^{-1} J\psi_0, J\psi_0 \rangle = \frac{1}{4\omega \|\varphi_\omega\|^2} \partial_\omega \|\varphi_\omega\|^2.$$

It is easy to see that the scaling arguments imply that $\varphi_\omega(x) = (1 - \omega^2)^{\frac{1}{p-1}} \varphi_0((1 - \omega^2)^{1/4} x)$, see (7.5), [26]. Thus,

$$\langle \mathcal{H}^{-1} J\psi_0, J\psi_0 \rangle = -\frac{\frac{1}{p-1} - \frac{d}{8}}{(1 - \omega^2)},$$

whence instability always occurs for $p \geq 1 + \frac{8}{d}$ or else, if

$$|\omega| \geq \sqrt{\frac{p-1}{4 - (p-1)(\frac{d}{2} - 1)}} = \omega_{p,d}^{beam}.$$

6. PROOF OF THEOREM 4

Before we start looking at the stability/instability of the KGZ system, let us settle a question that we have left open in Section 1.1.3, namely that that instability in the variables w and z (recall $w = z_x$) is equivalent. In fact, it is obvious that instabilities in z produces instability in w , by simply setting $z = e^{\lambda t} Z$, $w = e^{\lambda t} Z'$. Conversely, suppose we have constructed an unstable solution in terms of w , that is $w = e^{\lambda t} W$, $v = e^{\lambda t} V$, solve (8). In particular, W satisfies

$$(\lambda^2 - \partial_{xx})W = (\varphi_\omega \Re V)_{xx}$$

see (8). Thus $W = (\lambda^2 - \partial_{xx})^{-1} \partial_{xx}(\varphi_\omega \Re V)$, since $\lambda^2 - \partial_x^2$ is invertible. Thus, if we set $Z = (\lambda^2 - \partial_{xx})^{-1} \partial_x(\varphi_\omega \Re V)$, so that $W = Z'$, whence $v = e^{\lambda t} V$, $z = e^{\lambda t} Z$ is unstable in the sense that it satisfies (9). Thus, the instability of the KGZ system is equivalent to the instability of (9) and we concentrate on that.

We have already outlined the construction of the operators J, \mathcal{H} in Section 1.1.3. Thus, in order to apply Theorem 1, we need to verify that the conditions (12), (13), (14), (15) are met, after which we will compute the index $\omega^*(\mathcal{H})$. Note that (12) and (13) are obvious by inspection. It thus remains to verify (14) and (15).

We now turn to the properties (14) and (15), which need to be verified. This is accomplished in the following propositions. We start with H_-

Proposition 5. *The operator H_- , defined in (11) has a simple eigenvalue at zero, with eigenvector φ_ω . Moreover $H_-|_{\{\varphi_\omega\}^\perp} \geq \kappa^2 > 0$.*

The statement of this Proposition lists the well-known properties of L_- , the operator that arises in the linearization around the standard KdV soliton, so this is all well-known, see for example [41]. Indeed, one only needs to observe that since $\psi_\omega = -\frac{1}{2}\varphi_\omega^2$,

$$H_- = -\partial_{xx} + (1 - \omega^2) + \psi_\omega = -\partial_{xx} + (1 - \omega^2) - \frac{1}{2}\varphi_\omega^2.$$

Next, we tackle H_+ .

Proposition 6. *The self-adjoint operator H_+ defined in (11) has one simple eigenvalue at zero, with eigenvector $\begin{pmatrix} \varphi_\omega' \\ -\frac{\varphi_\omega^2}{2} \end{pmatrix}$ and one simple negative eigenvalue.*

We postpone the proof of Proposition 6 for the subsequent section, and we concentrate on the rest of the proof of Theorem 4. Based on Propositions 5 and 6, we conclude that the requirements of Theorem 1 are met. In fact, (14) is contained in the statements of the two

Propositions. Moreover, note that $\psi_0 = \|\varphi_\omega\|^{-1} \begin{pmatrix} 0 \\ 0 \\ \varphi_\omega \end{pmatrix}$, while $\psi_1 = c \begin{pmatrix} \varphi'_\omega \\ -\frac{\varphi_\omega^2}{2} \\ 0 \end{pmatrix}$ and hence

clearly $\langle \psi_1, J\psi_0 \rangle = C \langle \varphi_\omega, \varphi'_\omega \rangle = 0$, so (15) is satisfied as well.

It remains to apply the result. We have

$$\langle \mathcal{H}^{-1}[J\psi_0], J\psi_0 \rangle = \|\varphi_\omega\|^{-2} \left\langle H_+^{-1} \begin{pmatrix} \varphi_\omega \\ 0 \end{pmatrix}, \begin{pmatrix} \varphi_\omega \\ 0 \end{pmatrix} \right\rangle.$$

Thus, we are led to solve

$$H_+ \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \varphi_\omega \\ 0 \end{pmatrix}.$$

From the second equation, we have $-(\varphi_\omega f)' - g'' = 0$, whence $g' = -\varphi_\omega f$. Inserting this in the first equation yields

$$(40) \quad ((-\partial_x^2 + (1 - \omega^2) - \frac{3}{2}\varphi_\omega^2)f = \varphi_\omega.$$

Recall now the basic defining equation of φ_ω ,

$$(41) \quad -\varphi_\omega'' + (1 - \omega^2)\varphi_\omega - \frac{1}{2}\varphi_\omega^3 = 0$$

Taking a derivative with respect to ω in (41) yields

$$((-\partial_x^2 + (1 - \omega^2) - \frac{3}{2}\varphi_\omega^2)\partial_\omega \varphi = 2\omega\varphi_\omega.$$

Thus, the solution to (40) (which we know is unique, since $J\psi_0 \perp \text{Ker}(\mathcal{H})$) is $f = \frac{\partial_\omega \varphi}{2\omega}$. Thus

$$\langle \mathcal{H}^{-1}[J\psi_0], J\psi_0 \rangle = \|\varphi_\omega\|^{-2} \langle f, \varphi_\omega \rangle = \frac{\langle \partial_\omega \varphi, \varphi \rangle}{2\omega \|\varphi_\omega\|^2} = \frac{\partial_\omega [\|\varphi_\omega\|^2]}{4\omega \|\varphi_\omega\|^2}.$$

From the formula for $\varphi_\omega(y)$, we compute $\|\varphi_\omega\|^2 = 2\sqrt{1 - \omega^2} \int_{-\infty}^{\infty} \text{sech}^2(z) dz$, whence

$$\langle \mathcal{H}^{-1}[J\psi_0], J\psi_0 \rangle = -\frac{1}{4(1 - \omega^2)}.$$

This is clearly negative for all $\omega \in (-1, 1)$, and hence we always have stability for some values of ω . More precisely, the stability region is the solution of the following inequality

$$|\omega| \geq \frac{1}{2\sqrt{-\langle \mathcal{H}^{-1}[J\psi_0], J\psi_0 \rangle}} = \sqrt{1 - \omega^2}.$$

The solution to the last inequality is $1 > |\omega| \geq \frac{\sqrt{2}}{2}$, as is the statement of Theorem 4.

6.1. Proof of Proposition 6. First, we show the uniqueness of the eigenvalue at zero and we compute the eigenvector. We have $H_+ \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thus

$$\begin{aligned} (-\partial_x^2 + (1 - \omega^2) - \frac{\varphi_\omega^2}{2})f + \varphi_\omega g' &= 0 \\ -(\varphi_\omega f)' - g'' &= 0 \end{aligned}$$

From the second equation, we find (after integration in x) that $g' = -\varphi_\omega f$. Inserting this in the first equation yields

$$((-\partial_x^2 + (1 - \omega^2) - \frac{3}{2}\varphi_\omega^2)f = 0.$$

This is exactly the equation $L_+[f] = 0$, which again appears in the linearization of KdV around the standard soliton. It is well-known that the only solution to it (as can be seen by just differentiating the defining equation (2) in x) is $f = \varphi'_\omega$. Turning back to g , we find that $g' = -\varphi_\omega \varphi'_\omega$, implying $g = -\frac{\varphi_\omega^2}{2}$. Thus, H_+ has an unique eigenvector at zero, namely $\begin{pmatrix} \varphi'_\omega \\ -\frac{\varphi_\omega^2}{2} \end{pmatrix}$.

We now show that there is a simple negative eigenvalue for H_+ . We use the approach of the proof in Proposition 9 in our previous paper [39]. We need to show that there is an unique $a > 0$, so that the eigenvalue problem

$$(42) \quad H_+ \begin{pmatrix} f \\ g \end{pmatrix} = -a^2 \begin{pmatrix} f \\ g \end{pmatrix}$$

has an unique (up to multiplicative constant) solution $\begin{pmatrix} f \\ g \end{pmatrix}$ and for all other values of a , (42) has only the trivial solution. Writing (42) in detail,

$$\begin{cases} -f'' + (1 - \omega^2)f - \frac{1}{2}\varphi^2 f + \varphi g' = -a^2 f \\ -g'' - (\varphi f)' = -a^2 g \end{cases}$$

From the second equation, we conclude $g = \partial_x(a^2 - \partial_x^2)^{-1}[\varphi f]$. Thus,

$$g' = \partial_x^2(a^2 - \partial_x^2)^{-1}[\varphi f] = -\varphi f + a^2(a^2 - \partial_x^2)^{-1}[\varphi f].$$

Plug this in the first equation to obtain

$$(43) \quad -f'' + (1 + a^2 - \omega^2)f - \frac{3}{2}\varphi^2 f + a^2\varphi(a^2 - \partial_x^2)^{-1}[\varphi f] = 0$$

To recapitulate, we have shown that the eigenvalue problem (42) is equivalent to the solvability of (43). Introduce the self-adjoint operator

$$M_a := -\partial_x^2 + (1 + a^2 - \omega^2) - \frac{3}{2}\varphi^2 + a^2\varphi(a^2 - \partial_x^2)^{-1}[\varphi].$$

We need to show that there exists an unique a_0 , so that $0 \in \sigma(M_{a_0})$ and moreover, the eigenvalue zero is a simple eigenvalue for M_{a_0} . Before we proceed with the proof of this, let us establish the following

Claim 1. *Let $a \geq b \geq 0$. Then $M_a \geq M_b + (a^2 - b^2)Id \geq M_b$.*

Proof. (Claim) Since

$$M_a f - M_b f = (a^2 - b^2)f + a^2\varphi(a^2 - \partial_x^2)^{-1}[\varphi f] - b^2\varphi(b^2 - \partial_x^2)^{-1}[\varphi f]$$

The inequality easily reduces to checking that for all test functions f and for $a \geq b \geq 0$,

$$a^2 \langle (a^2 - \partial_x^2)^{-1}[\varphi f], \varphi f \rangle \geq b^2 \langle (b^2 - \partial_x^2)^{-1}[\varphi f], \varphi f \rangle.$$

Setting $h = \varphi f$, and by using the Fourier transform representation of $(a^2 - \partial_x^2)^{-1}$

$$\begin{aligned} a^2 \langle (a^2 - \partial_x^2)^{-1}[\varphi f], \varphi f \rangle &= \int \frac{a^2}{a^2 + 4\pi^2\xi^2} |\hat{h}(\xi)|^2 d\xi \geq \int \frac{b^2}{b^2 + 4\pi^2\xi^2} |\hat{h}(\xi)|^2 d\xi = \\ &= b^2 \langle (b^2 - \partial_x^2)^{-1}[\varphi f], \varphi f \rangle, \end{aligned}$$

where we have used the elementary inequality $\frac{a^2}{a^2 + 4\pi^2\xi^2} \geq \frac{b^2}{b^2 + 4\pi^2\xi^2}$. \square

Now that we have established the Claim, we are ready to show that there exist an unique a_0 , so that $0 \in \sigma(M_{a_0})$. To that end, define

$$\lambda_0(a) = \inf \sigma(M_a) = \inf_{\|f\|=1} \langle M_a f, f \rangle.$$

This is clearly a continuous function, which is also increasing by Claim 1. It is also immediately clear that $\lim_{a \rightarrow \infty} \lambda_0(a) = \infty$. Indeed, note that φ is a bounded function and $\|a^2 \varphi(a^2 - \partial_x^2)^{-1}[\varphi \cdot]\|_{L^2 \rightarrow L^2} \leq \|\varphi\|_{L^\infty}^2$. Regarding $\lambda_0(0)$, consider $M_0 = -\partial_x^2 + (1 - \omega^2) - \frac{3}{2}\varphi^2$. Recall that this is exactly the operator L_+ , which has one negative eigenvalue, $\lambda_0(0)$ in our notations. Hence $\lambda_0(0) < 0$.

By our analysis of the function $\lambda_0(a)$, we can conclude that there is an unique $a_0 > 0$, so that $\lambda_0(a_0) = 0$. That is, by the equivalence of (43) and (42), there is unique a_0 , so that the eigenvalue problem (42) has a non-trivial solution. It now remains only to show that the eigenvalue 0 in the spectrum of M_{a_0} is simple. This may be shown in a number of ways, but here is an outline. Recall that $M_0 = L_+$ has a single simple negative eigenvalue, corresponding to an eigenvector say ϕ_0 . Thus, $M_0|_{\{\phi_0\}^\perp} \geq 0$. By Claim 1 and the Courant maxmin principle, we have that the second eigenvalue $\lambda_1(M_{a_0})$ satisfies

$$\lambda_1(M_{a_0}) = \sup_{z \neq 0} \inf_{\|u\|=1, u \perp z} \langle M_{a_0} u, u \rangle \geq \inf_{\|u\|=1, u \perp \phi_0} \langle M_{a_0} u, u \rangle \geq a_0^2 + \inf_{\|u\|=1, u \perp \phi_0} \langle M_0 u, u \rangle \geq a_0^2.$$

Thus, $\lambda_1(M_{a_0}) \geq a_0^2 > 0$ and $\lambda_0(a_0) = 0$ is simple. Theorem 4 is proved.

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