(1) Consider the linear transport equation

$$u_t + \frac{1}{\sin x} u_x = 0$$

Find and sketch the characteristic curves, write a formula for the general solution and find the solution to the initial value problem with $u(x, 0) = \sin^2 x$

Characteristic curves are given by:

$$\frac{dx}{dt} = \frac{1}{\sin x}$$

$$\int \sin x \, dx = \int dt$$

$$-\cos x = t + K$$

$$x = \cos^{-1}(-t-K)$$

$\xi = -t - \cos x$ and the general solution is given by $\boxed{u(t, x) = v(t + \cos x)}$

To find the function $\underline{\underline{v}}$, use the initial condition:

$$v(t + \cos x)\Big|_{t=0} = v(\cos x) = \sin^2 x = 1 - \cos^2 x$$

$$\Rightarrow \quad v(y) = 1 - y^2$$

Then the solution of the IVP is

$$u(t, x) = v(t + \cos x) = 1 - (t + \cos x)^2$$

(2) Use D'Alembert's formula to solve the one dimensional wave equation

$$u_{tt} = 4u_{xx},$$

subject to initial conditions

$$u(x,0) = \sin x, \frac{\partial u}{\partial t}(x,0) = \cos x$$

$$u(t,x) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$$

$$u(t,x) = \frac{\sin(x+2t)+\sin(x-2t)}{2} + \frac{1}{4}\int_{x-2t}^{x+2t} \cos z\,dz$$

$$= \frac{\sin(x+2t)}{2} + \frac{\sin(x-2t)}{2} + \frac{\sin(x+2t)}{4} - \frac{\sin(x-2t)}{4}$$

$$= \frac{3}{4}\sin(x+2t) + \frac{1}{4}\sin(x-2t)$$

(3) Solve the Klein-Gordon equation with Dirichlet boundary conditions. That is, using the method of separation of variables, find the solution of

$$\begin{cases} u_{tt} - u_{xx} + u = 0, \ t > 0, \ 0 < x < L \\ u(0,t) = u(L,t) = 0, \ t > 0 \\ u(x,0) = f(x), u_t(x,0) = g(x) \end{cases}$$

Suppose $u(t,x) = T(t)\cdot X(x)$, then

$$T''\cdot X - T\cdot X'' + T\cdot X = 0 \quad \text{in } 0 < x < L$$

$$\frac{T''}{T} + 1 = \frac{X''}{X} = \lambda$$

$$\begin{cases} X'' - \lambda X = 0 \\ X(0) = 0 \\ X(L) = 0 \end{cases} \quad \text{and} \quad T'' + (1-\lambda)T = 0$$

$\lambda = \mu^2$ and $\lambda = 0$ give trivial solutions only.

$\lambda = -\mu^2$ gives $X(x) = C_1\cdot\cos\mu x + C_2\cdot\sin\mu x$

$X(0) = 0$ means $C_1 = 0$

$X(L) = \sin\mu L = 0$ gives $\boxed{\mu_n = \frac{n\pi}{L}}$

and $\boxed{X_n(x) = \sin\left(\frac{n\pi}{L}\cdot x\right)}$

$$\Downarrow$$

$$T'' + \left(1 + \frac{n^2\pi^2}{L^2}\right)T = 0$$

$T_n = A_n\cos K_n t + B_n\sin K_n t$ with $\boxed{K_n = 1 + \frac{n^2\pi^2}{L^2}}$

Normal modes:

$$u_n(t,x) = (A_n\cos K_n t + B_n\sin K_n t)\cdot\sin\left(\frac{n\pi x}{L}\right)$$

Thus

$$u(t,x) = \sum_{n=1}^{\infty}(A_n\cos K_n t + B_n\sin K_n t)\sin\left(\frac{n\pi}{L}x\right) \quad \text{where}$$

$$u(0,x) = \sum_{n=1}^{\infty} A_n\sin\left(\frac{n\pi}{L}x\right) = f(x) \quad \text{and} \quad u_t(0,x) = g(x) = \sum_{n=1}^{\infty} B_n K_n\sin\left(\frac{n\pi}{L}x\right)$$

(4) Solve the non-homogeneous boundary value problem for the one dimensional heat equation on a bar with unit length and $c = 1$, for the following data:

$$u(0,t) = 100, u(1,t) = 0, u(x,0) = 30\sin(\pi x).$$

The steady state is given by $u_1(x) = -100x + 100$

We then have to solve:

$$\begin{cases} u_t = u_{xx} \\ u(0,t) = 0 \\ u(L,t) = 0 \\ u(x,0) = 30\sin(\pi x) + 100x - 100 \end{cases}$$

The solution of this IVP is given by

$$u_2(x,t) = \sum_{n=1}^{\infty} b_n \cdot e^{-n^2\pi^2 t} \cdot \sin(n\pi x)$$

where $b_n = 2\int_0^1 [30\sin(\pi x) + 100x - 100] \sin(n\pi x)\, dx$

The solution is then $u(x,t) = u_1(x) + u_2(x,t)$

$$u(x,t) = 100(1-x) + 30e^{-\pi^2 t}\sin(\pi x) - \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{e^{-n^2\pi^2 t}}{n} \cdot \sin(n\pi x)$$

where we have used that the sin Fourier series of the function $f(x) = x$ is given by

$$x = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \sin(n\pi x)$$

for $0 \le x < 1$