LINEAR STABILITY OF SOLITARY WAVES FOR THE ONE-DIMENSIONAL BENNEY-LUKE AND KLEIN-GORDON EQUATIONS

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The linear stability of the solitary waves for the one-dimensional Benney-Luke equation in the case of strong surface tension is investigated rigorously and the critical wave speeds are computed explicitly. For the Klein-Gordon equation, the stability of the traveling standing waves is considered and the exact ranges of the wave speeds and the frequencies needed for stability are derived. This is achieved via the abstract stability criteria recently developed by Stanislavova and Stefanov.

1. Introduction and Results

In the last two decades there has been considerable research on model water wave equations and the stability of their solitary waves. Among them, the Boussinesq type models such as [2], [3] and [5] describe small amplitude long waves in water of finite length. In this paper we will study the one-dimensional Benney-Luke equation and the one-dimensional Klein-Gordon equation. Our goal is twofold - we aim to illustrate the usefulness of the abstract criteria for stability developed in [15], [16] as well as to supplement with exact computation of the wave speeds the orbital stability results for Benney-Luke equation developed in [12]. For the Klein-Gordon model, our interest is in the traveling standing waves, which depend on two parameters - the wave speed c and the frequency ω. The spectral stability of these for particular ranges of the parameters is the goal of this investigation.

1.1. Klein-Gordon equation.

We consider the following Klein-Gordon equation

$$u_{tt} - u_{xx} + u - |u|^{p-1}u = 0, \tag{1}$$

where u is a complex-valued function. We are interested in the stability of traveling-standing wave solutions of (1), which are in the form

$$e^{i\omega t} e^{iq(x-ct)} \varphi_{\omega,c}(x-ct), \tag{2}$$

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for some real parameters ω, c, q and a real-valued function φ, which depends on these parameters. Plugging in the ansatz (2) in (1) yields the ODE

$$(c^2-1)\varphi'' + 2i(c(\omega+qc)-q)\varphi' + (q^2-(\omega+qc)^2+1)\varphi - \varphi^p = 0, \quad x \in \mathbf{R}^1$$

In order to have real solution φ, we must then require $c(\omega+qc)-q=0$ or equivalently,

$$q = \frac{\omega c}{1-c^2}.$$

The defining equation for φ becomes

$$-(1-c^2)\varphi'' + \frac{1-\omega^2-c^2}{1-c^2}\varphi - \varphi^p = 0, \quad x \in \mathbf{R}^1 \tag{3}$$

In order to have existence, we will be assuming that $1 > \omega^2 + c^2$. With this assumption in mind, we introduce the positive parameters

$$\mu := 1-c^2 > 0, \quad \nu := \frac{1-\omega^2-c^2}{1-c^2} > 0,$$

so that (3) now reads

$$-\mu\varphi'' + \nu\varphi - \varphi^p = 0, \quad x \in \mathbf{R}^1 \tag{4}$$

The solution φ has the form

$$\varphi_{c,\omega}(x) = \left(\frac{(p+1)(1-\omega^2-c^2)}{2(1-c^2)^2}\right)^{\frac{1}{p-1}} sech^{\frac{2}{p-1}}\left(\frac{p-1}{2}\sqrt{\frac{1-\omega^2-c^2}{1-c^2}}\; x\right),$$

which can be written in terms of the parameters μ and ν as

$$\varphi_{\mu,\nu}(x) = \nu^{\frac{1}{p-1}}\varphi_0\left(\frac{\sqrt{\nu}}{\sqrt{\mu}}x\right)$$

and φ_0 is the solution of the equation $-\varphi'' + \varphi - \varphi^p = 0$. We have the following result.

Theorem 1. *Let $\omega^2+c^2<1$. For $p \leq 5$, the traveling standing waves for the Klein-Gordon equation* (1) *are stable if and only if*

$$1 < p \leq 1 + \frac{4\omega^2}{1-c^2}$$

For $p > 5$, the waves are always unstable.

1.2. **The one-dimensional Benney-Luke equation.** We consider the following model

$$u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} + u_t u_{xx} + 2u_x u_{tx} = 0. \tag{5}$$

This equation is an approximation of the full water wave equations, formally valid for describing two-way propagation unlike the well studied KdV and BBM equations, which are only valid for waves propagating in one direction. Here the positive parameters a and b are related to the inverse Bond number τ via $a-b=\tau-\frac{1}{3}$ and two distinct cases can be considered. The case of strong surface tension corresponds to $a>b>0$ and the case of small or zero surface tension to $0<a<b$. In both cases the traveling waves that decay at $\pm\infty$ can be described explicitly using the Hamiltonian structure of the equation after

a simple change of variables $q = u_x, r = u_t$ (see [12]). In particular, for any wave speed $c > 0$, and $c^2 < \min(1, a/b)$ the one-dimensional Benney-Luke model has the solutions

$$q_c(x) = \frac{c^2 - 1}{c} sech^2 \left(\frac{1}{2} \sqrt{\frac{1 - c^2}{a - bc^2}} x \right).$$

Taking advantage of the Hamiltonian formulation and using the variational approach of Grillakis, Shatah and Strauss ([7]), Quintero was able to prove the orbital stability of the waves in the case $0 < c < 1 < \sqrt{a/b}$ (see [12]). In the case $0 < c < \sqrt{a/b} < 1$ he proved the following instability result: there exists a wave speed $0 < c_* < \sqrt{a/b}$ such that the waves are orbitally unstable for wave speeds in a neighborhood of c^*.

While the variational approach works well in the case of strong surface tension, it fails in the case of small or zero surface tension due to the indefiniteness of the energy-momentum functional whose critical points are the solitary waves. The best result in this case appeared recently in the excellent paper [11], where the nonlinear stability in the energy norm under the assumption of spectral stability for the linearization is established in the case $0 < a < b$ and for speeds satisfying $c^2 > 1$. The authors also used suitable comparison of a reduced resolvent operator with the resolvent operator for the KdV solitons to establish spectral stability for waves of small energy. To our knowledge, the question of spectral stability for generic waves in the case of small or zero surface tension is still open.

We consider waves with speeds $c^2 < \min\{1, \frac{a}{b}\}$ and prove the following main result.

Theorem 2. *Under the assumptions $0 < c < 1, c < \sqrt{a/b}$ there exists a critical wave speed $c^* = c^*(\frac{a}{b})$ such that the waves with wave speed $|c| < c^*$ are linearly stable, while the waves with wave speeds $|c| > c^*$ are linearly unstable. Thus the stability region in the two-dimensional parameter space $(\frac{a}{b}, c^2)$ is precisely the region $\frac{a}{b} > f(c^2)$, while in the small region $c^2 < \frac{a}{b} < f(c^2)$ the waves are linearly unstable, see Fig.1. In other words the threshold speed is given by $c^* = \sqrt{f^{-1}(z)}$, where*

$$f(z) = \frac{19z^3 + 2z^2 + 9z - (z-1)z\sqrt{73z^2 + 54z + 33}}{2\,(6z^2 + 3z + 6)}.$$

1.3. **Different notions of stability.** We will discuss briefly the notions of stability used in the literature for abstract second order in time nonlinear PDEs of the form

$$u_{tt} + \mathcal{L}_x u + N(u) = 0, \quad (t, x) \in \mathbf{R}^1_+ \times \mathbf{R}^d \text{ or } (t, x) \in \mathbf{R}^1 \times [-L, L], \tag{6}$$

where $\mathcal{L}_x$ is a given linear operator, acting on the x variable and $N(u)$ is the nonlinear term. Consider traveling waves, which are solutions in the form $\varphi(x + \vec{c}t)$ and satisfy the stationary PDE

$$\mathcal{L}_x \varphi + c^2 \varphi_{xx} + N(\varphi) = 0 \tag{7}$$

Next, take the ansatz $u = \varphi(x + \vec{c}t) + v(t, x + \vec{c}t)$, plug it into (6), take into account (7) and drop all quadratic and higher order terms in v. The result is the following linearized equation for the perturbation

$$v_{tt} + 2cv_{xt} + c^2 v_{xx} + \mathcal{L}_x v + N'(\varphi) v = 0 \tag{8}$$

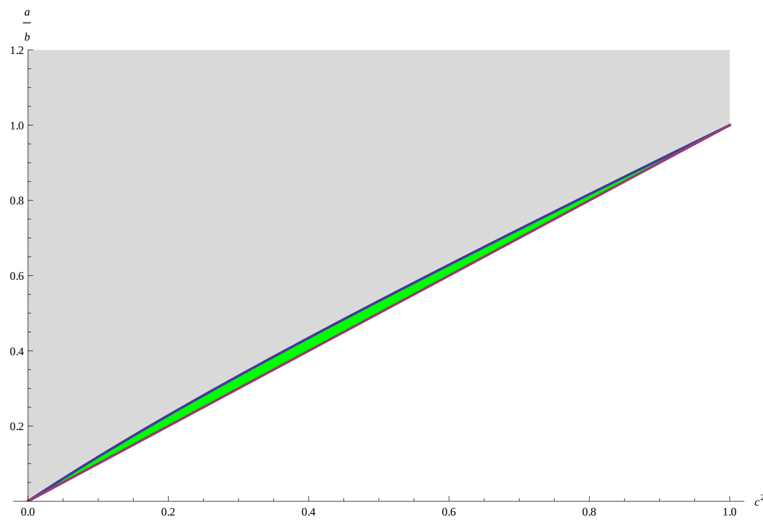

FIGURE 1. The waves are stable for values of the parameters a, b, c^2 in the large gray shaded region and unstable in the small green shaded region between the curves $\frac{a}{b} = f(c^2)$ and $\frac{a}{b} = c^2$

Introducing the operator $H_c = \mathcal{L}_x + c^2\partial_{xx} + N'(\varphi)$ allows us to consider the following problem

$$v_{tt} + 2cv_{xt} + H_c v = 0, \tag{9}$$

which determines the stability of the waves.

There are several notions of stability that are of interest. Consider an evolution equation $z_t = \mathcal{A}z$, where $\mathcal{A}$ is a closed operator generating C_0 semigroup. We say that the wave is *spectrally stable*, if $\sigma(\mathcal{A}) \subset \mathcal{Z}_- = \{\lambda : \Re\lambda \le 0\}$. We say that the same problem is *linearly stable*, if the solutions grow at infinity slower than any exponential, i.e. for every $\delta > 0$, $\lim_{t\to\infty} e^{-\delta t}\|z(t)\| = 0$. The relationship between spectral and linear stability is not trivial in the case of PDEs, but in the presence of the so-called spectral mapping theorem for the generator $\mathcal{A}$, these are equivalent and amount to lack of exponentially growing modes, that is solutions in the form $\mathcal{A}\psi = \lambda\psi$ for $\lambda : \Re\lambda > 0$.

Generally, spectral and linear stability is easier to check than nonlinear stability. On the other hand, we have two distinct notions of nonlinear stability - orbital and asymptotic stability. Assuming for simplicity that the only invariance of the system is translation, orbital stability requires that a solution for the full nonlinear equation that starts close to a traveling wave stay close for all times to a (time-dependent) translate of the starting wave. Asymptotic stability requires a bit more, namely that the perturbed profile will actually converge to a (time-dependent) translate of the wave.

There is a large body of literature that deals with this problem in various models. We would like to point out that the powerful methods of Grillakis-Shatah-Strauss, [7] reduce the problem of orbital stability to checking certain conditions on the linearized functionals. We also note that establishing orbital instability for a given problem seems to be harder and requires more problem specific efforts, [9].

We are interested in the question of linear stability of traveling waves φ_c for the Benney-Luke equation (5). We would like to determine for which values of $\vec{c}$ is the corresponding traveling wave $\varphi_{\vec{c}}$ linearly/spectrally stable? We will do this using the abstract criteria for linear stability/instability developed in [15]. In the next section we give a brief description of this theorem.

1.4. **Stability/instability results for quadratic pencils.** Consider linear, second-order in time equations in the general form

$$v_{tt} + 2\omega \mathcal{J} v_t + \mathcal{H} v = 0, (t,x) \in \mathbf{R}^1 \times \mathbf{R}^1 \text{ or } \mathbf{R}^1 \times [-L, L] \tag{10}$$

where $\mathcal{H} = \mathcal{H}_c$ is a self-adjoint operator acting on L^2, with domain $D(\mathcal{H})$, $\mathcal{J}$ is skew-adjoint and $\mathcal{H}$-bounded and ω is a real parameter. Note that it is better at this point to consider ω as an independent parameter, but in the applications $\omega = c$ is the wave speed of the traveling wave.

Definition 1. *We say that the quadratic pencil given by the couple (J, H) is spectrally unstable, if there exists an T periodic function $\psi \in D(\mathcal{H})$ and $\lambda : \Re\lambda > 0$, so that*

$$\lambda^2 \psi + 2\lambda J \psi + H\psi = 0. \tag{11}$$

Otherwise, we say that the quadratic pencil (J, H) is stable.

We now give precise statements of the results in [16], which characterize spectral stability in this context. Consider

$$\lambda^2 \psi + 2\lambda z J \psi + H\psi = 0 \tag{12}$$

Let $L^2 \otimes L^2 = X^+ \oplus X^-$, so that H acts invariantly on both $X^\pm$ and $J : X^\pm \to X^\mp$. We assume the following for the spectrum of H

$$\left\{ \begin{array}{l} H\phi = -\delta^2 \phi, H\mid_{\{\phi\}^\perp} \geq 0 \\ Ker[H] = span[\psi_0, \psi_1, \dots, \psi_n], \ \|\psi_j\| = 1, j = 0, \dots, n \\ \psi_0 \in X^-, \phi, \psi_1, \dots, \psi_n \in X^+. \end{array} \right. \tag{13}$$

In addition, we make the following assumptions:

$$\overline{Hu} = H\overline{u}, H^* = H, \tag{14}$$

$$\overline{Ju} = J\overline{u}, J^* = -J, J(H+A)^{-1} \in B(L^2), A >> 1 \tag{15}$$

$$\langle \psi_1, J\psi_0 \rangle = 0 \tag{16}$$

The following theorem is proved in [16]

Theorem 3. *Let H, J be a self-adjoint and anti-self adjoint operators respectively on a Hilbert space H, so that they satisfy the assumptions (13), (14), (15),* (16).

If $\langle H^{-1}[J\psi_0], J\psi_0 \rangle \geq 0$, one has a solution of (12) *for all values of z, that is instability in sense of Definition 1. Otherwise, supposing that $\langle H^{-1}[J\psi_0], J\psi_0 \rangle < 0$*

- *the problem* (12) *has solution, if z satisfies the inequality*

$$|z| < \frac{1}{2\sqrt{-\langle H^{-1}[J\psi_0], J\psi_0 \rangle}} =: z^*(H) \tag{17}$$

• *the problem* (12) *does not have solutions (i.e. stability), if* w *satisfies the reverse inequality*

$$|z| \geq z^*(H) \tag{18}$$

Thus, in order to determine the stability, one needs to compute the quantity $\langle \mathcal{H}^{-1}[\mathcal{J}\psi_0, \mathcal{J}\psi_0\rangle$ for the particular solution at hand.

2. Threshold speed computation for the Benney-Luke equation

2.1. Traveling waves and setup in the form of a quadratic pencil. Assume $b > a > 0$ and consider solitary wave of the form $\varphi(x - ct)$ of the Benney-Luke model (5), which satisfies the equation

$$(a - bc^2)\varphi'''' - (1 - c^2)\varphi'' - 3c\varphi'\varphi'' = 0. \tag{19}$$

We will use the function $q = \varphi_x$, which after one integration satisfies the equation

$$-(a - bc^2)q'' + (1 - c^2)q + \frac{3c}{2}q^2 = 0. \tag{20}$$

From this last equation one can obtain the explicit form of the solution in the case $a - bc^2 > 0$ and $c^2 < 1$, namely

$$q(x) = \frac{c^2 - 1}{c} sech^2 \left(\frac{1}{2}\sqrt{\frac{1 - c^2}{a - bc^2}} x \right).$$

Consider next the perturbed solution $u(x, t) = \varphi(x - ct) + v(t, x - ct)$ and the linearized equation for the perturbation $v(t, x - ct)$, given by the equation

$$(1 - b\partial_x^2)[v_{tt} - 2cv_{tx} + c^2 v_{xx}] + av_{xxxx} - v_{xx} - 3c\varphi' v_{xx} - 3c\varphi'' v_x + v_t\varphi'' + 2\varphi' v_{tx} = 0.$$

The last equation can be written in the form

$$B(v_{tt} - 2cv_{tx} + c^2 v_{xx}) + Jv_t + \mathcal{L}v = 0,$$

where

$$B = 1 - b\partial_x^2, Jv = v\varphi^{''} + 2\varphi' v_x, \quad \mathcal{L}v = av_{xxxx} - v_{xx} - 3c\partial_x(\varphi' v_x).$$

Since $B = 1 - b\partial_x^2$ is invertible, the change of variables $w = B^{1/2}v$ gives the equation

$$w_{tt} - 2cw_{tx} + c^2 w_{xx} + J_B w_t + \mathcal{L}_B w = 0,$$

with $J_B = B^{-1/2} J B^{-1/2}, \mathcal{L}_B = B^{-1/2}\mathcal{L}B^{-1/2}$. This puts the linearized equation in the desired form (see (9))

$$w_{tt} - 2c\mathcal{J}w_t + \mathcal{H}w = 0, \tag{21}$$

where the operators

$$\mathcal{J} = \partial_x - \frac{1}{2c}J_B, \quad \mathcal{H} = \mathcal{L}_B + c^2\partial_x^2 = B^{-1/2}\mathcal{L}B^{-1/2} + c^2\partial_x^2$$

have to satisfy the assumptions in Theorem 3, which we will check next.

Lemma 1. *The operator* $\mathcal{H}$ *with the natural domain* $Dom(\mathcal{H}) = H^2(\mathbf{R})$ *is self-adjoint, while the operator* $\mathcal{J}$ *with domain* $H^1(\mathbf{R})$ *is skew-adjoint.*

The self-adjointness of $\mathcal{H}$ is clear from the form of the operator. To prove the statement for $\mathcal{J}$ note that ∂_x is skew-adjoint and look at

$$\langle J_B w_1, w_2\rangle = \langle B^{-1/2} J B^{-1/2} B^{1/2} v_1, B^{1/2} v_2\rangle = \langle J v_1, v_2\rangle.$$

Next,

$$\begin{aligned}\langle J v_1, v_2\rangle &= \int v_1\varphi'' v_2 dx + 2\int v_1'\varphi' v_2 dx = \int v_1\varphi'' v_2 dx - 2\int v_1(\varphi'' v_2 + \varphi' v_2')dx = \\ &= -\int v_1\varphi'' v_2 dx - 2\int v_1\varphi' v_2' = -\langle v_1, J v_2\rangle.\end{aligned}$$

In the next lemma, we collect the required information about the spectrum of $\mathcal{H}$.

Lemma 2. *The operator $\mathcal{H}$ has exactly one simple negative eigenvalue, a simple eigenvalue at zero and the absolutely continuous spectrum is the semiaxis $[0,\infty)$.*

In the proof of this lemma, we will use the operator

$$L = -(a-bc^2)\partial_x^2 + (1-c^2) + 3c\varphi'$$

to simplify our notations. Notice that for the function $q(x)=\varphi'(x)$ defined above one has from (19) that

$$L(q_x) = -(a-bc^2)q_{xxx} + (1-c^2)q_x + 3cqq_x = 0.$$

Thus the function q_x is an eigenvector for the eigenvalue zero of the operator L. In addition since q_x changes sign exactly once, by Sturm-Liouville theory ([17]) the operator L has exactly one simple negative eigenfunction, which we will denote by ϕ. Thus we have that

$$L\phi = -\sigma^2\phi.$$

Going back to the operator $\mathcal{H}$, we can rewrite it in the form

$$\begin{aligned}\mathcal{H} &= \mathcal{L}_B + c^2\partial_x^2 = \mathcal{L}_B + B^{-1/2}\partial_x c^2(1-b\partial_x^2)B^{-1/2}\partial_x = \\ &= -B^{-1/2}\partial_x(-(a-bc^2)\partial_x^2 + (1-c^2) + 3c\varphi')\partial_x B^{-1/2} = -B^{-1/2}\partial_x L\partial_x B^{-1/2}.\end{aligned}$$

Thus $\mathcal{H}(B^{1/2}q) = -B^{-1/2}\partial_x L(q_x) = 0$, which shows that zero is an eigenvalue of $\mathcal{H}$ with an eigenvector

$$\psi_0 = B^{1/2}q.$$

We can easily convince ourselves that $dim(Ker[\mathcal{H}])=1$. Indeed, if $H[\Psi]=0$, it follows that $B^{-1/2}\partial_x\Psi \in Ker[L] = span\{q'\}$. Thus, $\Psi = const.B^{1/2}q$ and we obtain a multiple of the previous eigenfunction

Next we will show that $\mathcal{H}$ has at least one negative eigenvalue. Assume $\mathcal{H}$ has no negative spectrum and compute for $h = B^{1/2}q_x$,

$$\begin{aligned}\langle\mathcal{H}h, h\rangle &= \langle -B^{-1/2}\partial_x L\partial_x B^{-1/2}h, h\rangle = \langle L\partial_x B^{-1/2}h, \partial_x B^{-1/2}h\rangle = \\ &= \langle Lq_{xx}, q_{xx}\rangle = 0.\end{aligned}$$

To verify the last statement, differentiate the equality

$$0 = L(q_x) = -(a-bc^2)q_{xxx} + (1-c^2)q_x + 3cqq_x$$

to get that

$$L(q_{xx}) = -(a-bc^2)q_{xxxx} + (1-c^2)q_{xx} + 3cqq_{xx} = (L(q_x))' - 3c(q_x)^2 = -3c(q_x)^2$$

and from that

$$\langle Lq_{xx}, q_{xx}\rangle = -3c\int (q_x)^2 q_{xx} dx = 0.$$

But according to our assumption $\mathcal{H} \geq 0$. Thus $\langle \mathcal{H}h, h\rangle = 0$ is only possible if $\mathcal{H}h = 0$ or equivalently $L(\partial_x B^{-1/2}h) = L(q_{xx}) = 0$. On the other hand,

$$\dim(KerL) = 1, q_x \in KerL$$

and $q_{xx} \perp q_x$, whence $q_{xx} \notin KerL$, which gives a contradiction.

Thus we have shown that $\mathcal{H}$ has at least one negative eigenvalue $\lambda_0(\mathcal{H})$. Since the bottom of the spectrum is negative and there is a simple eigenvalue at zero, we can conclude that $\lambda_1(\mathcal{H}) \leq 0$. On the other hand, one has the following estimate, which shows that $\lambda_1(\mathcal{H}) \geq 0$ and gives $\lambda_1(\mathcal{H}) = 0$.

$$\begin{aligned}
\lambda_1(\mathcal{H}) &= \sup_g \inf_{u \perp g, \|u\|=1} \langle -B^{-1/2}\partial_x L \partial_x B^{-1/2}u, u\rangle \geq \\
&\geq \inf_{B^{-1/2}u \perp \psi_0'} \langle L\partial_x B^{-1/2}u, \partial_x B^{-1/2}u\rangle = \\
&= \inf_{\partial_x B^{-1/2}u \perp \psi_0} \langle L\partial_x B^{-1/2}u, \partial_x B^{-1/2}u\rangle \geq 0.
\end{aligned}$$

This line of reasoning establishes that $\mathcal{H}$ has one simple and negative eigenvalue and a simple eigenvalue at zero. Next, we compute the absolutely continuous spectrum of $\mathcal{H}$. We will show that $\sigma_{a.c.}[\mathcal{H}] = [0, \infty)$. Indeed, by Weyl's theorem,

$$\sigma_{a.c.}[\mathcal{H}] = \sigma_{a.c.}[\mathcal{H}_0] = \sigma[-B^{-1/2}\partial_x[-(a - bc^2)\partial_x^2 + (1 - c^2)]\partial_x B^{-1/2}].$$

By Fourier transform, the said spectrum is equal to the range of its symbol, namely

$$\sigma_{a.c.}[\mathcal{H}] = Range[\xi \to \frac{\xi^2[(a - bc^2)\xi^2 + 1 - c^2]}{(1 + b\xi^2)}] = [0, \infty).$$

This is a bit of an unfortunate situation, because Theorem 3 requires that the operator $\mathcal{H}$ has spectral gap, that is $\sigma_{a.c.}[\mathcal{H}] \subset (\sigma^2, \infty)$ for some strictly positive σ, while clearly $\mathcal{H}$ fails that. We can however still apply these results to a (family) of perturbed problems, whence the result for $\mathcal{H}$ will follow as a limiting case. More precisely, consider for $\varepsilon > 0$, the self-adjoint operator

$$\mathcal{H}_\varepsilon := -B^{-1/2}(\partial_x + \varepsilon\partial_x|\partial_x|^{-1})L(\partial_x + \varepsilon\partial_x|\partial_x|^{-1})B^{-1/2}.$$

Here, the operator $\partial_x|\partial_x|^{-1}$ acting on test functions as follows

$$\mathcal{F}[\partial_x|\partial_x|^{-1}f](\xi) = i\frac{\xi}{|\xi|}\hat{f}(\xi) = isgn(\xi)\hat{f}(\xi)$$

is the well-known Hilbert transform, which is a skew-symmetric operator. Now Weyl's theorem applied to $\mathcal{H}_\varepsilon$ yields the following formula for the a.c. spectrum of $\mathcal{H}_\varepsilon$ now gives

$$\sigma_{a.c.}[\mathcal{H}_\varepsilon] = Range[\xi \to \frac{(\xi + \varepsilon sgn(\xi))^2[(a - bc^2)\xi^2 + 1 - c^2]}{(1 + b\xi^2)}] = [\varepsilon^2(1 - c^2), \infty).$$

Furthermore, since $\mathcal{H}$ has a negative eigenvalue, the same will be true for all $\mathcal{H}_\varepsilon$, provided ε is small enough. Indeed, taking the (smooth) eigenvector, say $\tilde{\phi}$ of $\mathcal{H}$, $\mathcal{H}[\tilde{\phi}] = -\delta^2\tilde{\phi}$, we have

$$\begin{aligned}\left\langle \mathcal{H}_\varepsilon\tilde{\phi}, \tilde{\phi}\right\rangle &= \left\langle \mathcal{H}\tilde{\phi}, \tilde{\phi}\right\rangle + 2\varepsilon\left\langle L(\partial_x + \varepsilon\partial_x|\partial_x|^{-1})B^{-1/2}\tilde{\phi}, B^{-1/2}(\partial_x|\partial_x|^{-1})\tilde{\phi}\right\rangle = \\ &= -\delta^2 + O(\varepsilon) < -\frac{\delta^2}{2},\end{aligned}$$

for all small enough ε and hence $\mathcal{H}_\varepsilon$ has a negative eigenvalue. Similarly to the conclusion $\lambda_1(\mathcal{H}) \geq 0$, we have that $\lambda_1(\mathcal{H}_\varepsilon) \geq 0$. Regarding the eigenvalue at zero for $\mathcal{H}_\varepsilon$, it is easily verifiable that the smooth and decaying function (recall q is even)

$$\chi_\varepsilon(x) := 2\int_0^\infty \frac{\xi}{\xi + \varepsilon}\hat{q}(\xi)\cos(2\pi x\xi)d\xi.$$

satisfies $(\partial_x + \varepsilon\partial_x|\partial_x|^{-1})\chi_\varepsilon = \partial_x q$. Thus $\mathcal{H}_\varepsilon[B^{1/2}\chi_\varepsilon] = 0$ and zero is a simple eigenvalue with an eigenvector $B^{1/2}\chi_\varepsilon$, so $\lambda_1(\mathcal{H}_\varepsilon) = 0$. Based on the observations that we have made for $\mathcal{H}_\varepsilon$, we can apply the Theorem 3 to it. For technical reasons, we need to also slightly change the definition of J as well. More precisely, let

$$J^\varepsilon v := v\chi_\varepsilon' + 2\chi_\varepsilon v_x,\ J_B^\varepsilon := B^{-1/2}J^\varepsilon B^{-1/2},\ \mathcal{J}^\varepsilon := \partial_x - \frac{1}{2c}J_B^\varepsilon$$

Note that J^ε differs slightly from J in that we have replaced q by $\chi_\varepsilon = (1 + \varepsilon|\partial_x|^{-1})^{-1}q$. We now consider the following quadratic pencil problem

$$w_{tt} - 2c\mathcal{J}^\varepsilon w_t + \mathcal{H}_\varepsilon w = 0. \tag{22}$$

We need to however relate to the instability region of the eigenvalue problem (21) to the instability region for (22). Recall the following proposition, which appears in [16].

Proposition 1. *(Proposition 2, [16]) Assume that $\mathcal{H}$ satisfies $\sigma_{a.c.}[\mathcal{H}] = [0,\infty)$, but otherwise satisfies the structural assumptions above with say $\mathcal{H}\tilde{\phi} = -\delta^2\tilde{\phi}$. Then the eigenvalue problem (21) has solution (i.e. instability) if and only if the function*

$$\lambda \to \mathcal{G}(\omega,\lambda) := \left\langle [\mathcal{H} + \lambda^2 + 2\omega\lambda P_0\mathcal{J}P_0]^{-1}[\mathcal{J}\tilde{\phi}], \mathcal{J}\tilde{\phi}\right\rangle + \frac{\lambda^2 - \delta^2}{4\omega^2\lambda^2},$$

vanishes for some $\lambda_0 > 0$. Here $P_0 f := f - \left\langle f, \tilde{\phi}\right\rangle\tilde{\phi}$ is the projection onto $\{\tilde{\phi}\}^\perp$.

Note: The Proposition 1 still holds, even for operators $\mathcal{H}$ without a spectral gap, basically because the function $\mathcal{G}$ is well-defined for $\lambda > 0$. Theorem 3 follows by analyzing the behavior of $\mathcal{G}$ close to $\lambda = 0$, but in order to compute the limit $\lim_{\lambda\to 0}\mathcal{G}(\omega,\lambda)$ one needs to have the invertibility of $\mathcal{H}$ on $\{Ker\ \mathcal{H}\}^\perp$, which clearly fails if the a.c. spectrum touches the origin.

We can nevertheless infer the properties that we need for $\mathcal{H}$ by approximating by $\mathcal{H}_\varepsilon$. First, we need the following intuitive statement

Lemma 3. *For the operator $\mathcal{H}$, $\mathcal{H}_\varepsilon$, with $\mathcal{H}\tilde{\phi} = -\delta^2\tilde{\phi}$, $\mathcal{H}_\varepsilon\tilde{\phi}_\varepsilon = -\delta_\varepsilon^2\tilde{\phi}_\varepsilon$ and $\|\tilde{\phi}\| = \|\tilde{\phi}_\varepsilon\| = 1$, we have*

$$\|\tilde{\phi} - \tilde{\phi}_\varepsilon\|_{L^2}^2 + |\delta^2 - \delta_\varepsilon^2| \leq C\varepsilon.$$

Proof. The estimate for the eigenvalues follows as before, since

$$-\delta_\varepsilon^2 = \inf_{z:\|z\|=1} \langle \mathcal{H}_\varepsilon z, z\rangle \le \left\langle \mathcal{H}_\varepsilon \tilde{\phi}, \tilde{\phi} \right\rangle = -\delta^2 + O(\varepsilon).$$

The reverse inequality holds by reversing the roles of δ_ε^2 and δ^2. Similar approach works for the eigenfunctions. Indeed, since we have

$$(\mathcal{H} + \delta^2)\tilde{\phi}_\varepsilon = O(\varepsilon),$$

we conclude

$$O(\varepsilon) = \left\langle (\mathcal{H} + \delta^2)\tilde{\phi}_\varepsilon, \tilde{\phi}_\varepsilon \right\rangle = \left\langle (\mathcal{H} + \delta^2)[\tilde{\phi}_\varepsilon - \left\langle \tilde{\phi}_\varepsilon, \tilde{\phi} \right\rangle \tilde{\phi}], \tilde{\phi}_\varepsilon - \left\langle \tilde{\phi}_\varepsilon, \tilde{\phi} \right\rangle \tilde{\phi} \right\rangle.$$

But now, $\tilde{\phi}_\varepsilon - \left\langle \tilde{\phi}_\varepsilon, \tilde{\phi} \right\rangle \tilde{\phi} \in \{\tilde{\phi}\}^\perp$, whence

$$O(\varepsilon) \ge \delta^2 \|\tilde{\phi}_\varepsilon - \left\langle \tilde{\phi}_\varepsilon, \tilde{\phi} \right\rangle \tilde{\phi}\|^2,$$

which implies that $\|\tilde{\phi}_\varepsilon - \tilde{\phi}\|^2 = O(\varepsilon)$. □

Based on Lemma 3, it is clear that the Evans-like function associated to $\mathcal{H}_\varepsilon$

$$\mathcal{G}^\varepsilon(\omega, \lambda) := \left\langle [\mathcal{H}_\varepsilon + \lambda^2 + 2\omega\lambda P_0^\varepsilon \mathcal{J}^\varepsilon P_0^\varepsilon]^{-1}[\mathcal{J}^\varepsilon \tilde{\phi}_\varepsilon], \mathcal{J}^\varepsilon \tilde{\phi}_\varepsilon \right\rangle + \frac{\lambda^2 - \delta_\varepsilon^2}{4\omega^2\lambda^2},$$

converges to $\mathcal{G}(\omega, \lambda)$ for all fixed ω and $\lambda > 0$ as $\varepsilon \to 0+$ and therefore *the instability of the quadratic pencil is equivalent to the existence of $\lambda_0(\varepsilon) > 0$, so that $\mathcal{G}^\varepsilon(\omega, \lambda_0(\varepsilon)) = 0$ for all ε small enough.* Thus, according to Theorem 3, the instability of $\mathcal{H}_\varepsilon$ is decided based on the quantity $\langle \mathcal{H}_\varepsilon^{-1} \mathcal{J}^\varepsilon [B^{1/2}\chi_\varepsilon], \mathcal{J}^\varepsilon B^{1/2}\chi_\varepsilon \rangle$.

2.2. **Threshold speed computation.**

We have

$$\begin{aligned}
&\left\langle \mathcal{H}_\varepsilon^{-1} \mathcal{J}^\varepsilon [B^{\frac{1}{2}}\chi_\varepsilon], \mathcal{J}^\varepsilon B^{\frac{1}{2}}\chi_\varepsilon \right\rangle = \\
= \;& \left\langle (-B^{\frac{1}{2}}(\partial_x + \varepsilon\partial_x|\partial_x|^{-1})L(\partial_x + \varepsilon\partial_x|\partial_x|^{-1})B^{\frac{1}{2}})^{-1}\mathcal{J}^\varepsilon[B^{\frac{1}{2}}\chi_\varepsilon], \mathcal{J}^\varepsilon[B^{\frac{1}{2}}\chi_\varepsilon] \right\rangle \\
= \;& -\left\langle ((\partial_x + \varepsilon\partial_x|\partial_x|^{-1})L(\partial_x + \varepsilon\partial_x|\partial_x|^{-1}))^{-1}B^{\frac{1}{2}}\mathcal{J}^\varepsilon[B^{\frac{1}{2}}\chi_\varepsilon], B^{\frac{1}{2}}\mathcal{J}^\varepsilon[B^{\frac{1}{2}}\chi_\varepsilon] \right\rangle.
\end{aligned}$$

Note that

$$\begin{aligned}
B^{\frac{1}{2}}\mathcal{J}^\varepsilon B^{\frac{1}{2}}\chi_\varepsilon &= \frac{1}{2c}(2c\partial_x B - J^\varepsilon)\chi_\varepsilon = \frac{1}{2c}[2cB\partial_x\chi_\varepsilon' - (\chi_\varepsilon\chi_\varepsilon' + 2\chi_\varepsilon\chi_\varepsilon')] = \\
&= \frac{1}{2c}\partial_x[2cB\chi_\varepsilon - \frac{3}{2}\chi_\varepsilon^2].
\end{aligned}$$

Thus, we may compute

$$(\partial_x + \varepsilon\partial_x|\partial_x|^{-1})^{-1}[B^{\frac{1}{2}}\mathcal{J}^\varepsilon B^{\frac{1}{2}}\chi_\varepsilon] = \frac{1}{2c}(1 + \varepsilon|\partial_x|^{-1})^{-1}[2cB\chi_\varepsilon - \frac{3}{2}\chi_\varepsilon^2],$$

where the operator $(1+\varepsilon|\partial_x|^{-1})^{-1}$ acts by the multiplier $\frac{2\pi|\xi|}{2\pi|\xi|+\varepsilon}$. Thus, we have the formula

$$\left\langle \mathcal{H}_\varepsilon^{-1}\mathcal{J}^\varepsilon[B^{\frac{1}{2}}\chi_\varepsilon], \mathcal{J}^\varepsilon B^{\frac{1}{2}}\chi_\varepsilon \right\rangle =$$

$$= \frac{1}{4c^2}\left\langle L^{-1}[(1+\varepsilon|\partial_x|^{-1})^{-1}[2cB\chi_\varepsilon - \frac{3}{2}\chi_\varepsilon^2], (1+\varepsilon|\partial_x|^{-1})^{-1}[2cB\chi_\varepsilon - \frac{3}{2}\chi_\varepsilon^2]\right\rangle.$$

Note that the last expression is well-defined, since $Ker(L) = span\{q_x\}$, where q_x is an odd function, whereas $(1+\varepsilon|\partial_x|^{-1})^{-1}[2cB\chi_\varepsilon - \frac{3}{2}\chi_\varepsilon^2]$ is clearly even and hence perpendicular to $Ker(L)$. In addition, since for an arbitrary L^2 function g, one has

$$\lim_{\varepsilon\to 0}\|(1+\varepsilon|\partial_x|^{-1})^{-1}g - g\|_{L^2} = 0,$$

we may conclude that

$$\lim_{\varepsilon\to 0}\|(1+\varepsilon|\partial_x|^{-1})^{-1}[2cB\chi_\varepsilon - \frac{3}{2}\chi_\varepsilon^2] - [2cBq - \frac{3}{2}q^2]\|_{L^2} = 0$$

Hence the stability depends on the quantity

$$M(a,b,c) := \frac{1}{4c^2}\langle L^{-1}(2cBq - \frac{3}{2}q^2), 2cBq - \frac{3}{2}q^2\rangle.$$

in the way described in Theorem 3, that is stability occurs for all values of c so that

$$|c| \geq \begin{cases} +\infty & M(a,b,c) \geq 0 \\ \frac{1}{2\sqrt{-M(a,b,c)}} & M(a,b,c) < 0. \end{cases}$$

Since $Bq = q - bq''$, we need to compute $L^{-1}(q), L^{-1}(q^2), L^{-1}(q'')$. We will use the fact that $q(x) = \frac{c^2-1}{c}sech^2(\frac{1}{2}\sqrt{\frac{1-c^2}{a-bc^2}})$ satisfies equation (20) to do that. Define

$$\mu = \frac{1-c^2}{a-bc^2}, \quad \lambda = \frac{c}{a-bc^2}$$

to get that $q(x,\lambda,\mu) = -\frac{\mu^2}{\lambda}sech^2(\frac{\mu}{2}x)$ and

$$L(q) = -(a-bc^2)q'' + (1-c^2)q + \frac{3c}{2}q^2 = (a-bc^2)(-q'' + \mu q + \frac{3}{2}\lambda q) = (a-bc^2)\tilde{L}(q)$$

Thus for any function v

$$L^{-1}(v) = \frac{1}{a-bc^2}\tilde{L}^{-1}(v)$$

and we proceed to compute $\tilde{L}^{-1}$. We will differentiate in λ the equation

$$-q'' + \mu^2 q + \frac{3}{2}\lambda q^2 = 0$$

to get that $\tilde{L}(\partial_\lambda q) + \frac{3}{2}q^2 = 0$ and $\tilde{L}^{-1}(q^2) = -\frac{2}{3}\partial_\lambda q$. Thus

$$L^{-1}(q^2) = \frac{-\frac{2}{3}\partial_\lambda q}{a-bc^2} \tag{23}$$

$$L^{-1}(q) = \frac{-\frac{1}{2\mu}\partial_\mu q}{a-bc^2} \tag{24}$$

Finally, use that $q'' = \mu^2 q + \frac{3}{2}q^2$ to compute that

$$L^{-1}(q'') = \frac{-1}{a-bc^2}\left(\frac{\mu}{2}\partial_\mu q + \lambda\partial_\lambda q\right) \tag{25}$$

Going back to the computation of

$$\begin{aligned} M(a,b,c) &= \frac{1}{4c^2}\langle L^{-1}(2cq - 2cbq'' - \frac{3}{2}q^2), 2cq - 2bcq'' - \frac{3}{2}q^2\rangle = \\ &= \frac{1}{4c^2(a-bc^2)}\langle -\frac{c}{\mu}\partial_\mu q + \partial_\lambda q + bc\mu\partial_\mu q + 2bc\lambda\partial_\lambda q, 2cq - 2bcq'' - \frac{3}{2}q^2\rangle = \\ &= \frac{1}{4c^2(a-bc^2)}\langle (bc\mu - \frac{c}{\mu})\partial_\mu q + (1+2bc\lambda)\partial_\lambda q, 2cq - 2bcq'' - \frac{3}{2}q^2\rangle = \\ &= \frac{1}{4c^2(a-bc^2)}\int_{-\infty}^{\infty}[(bc\mu - \frac{c}{\mu})\partial_\mu q + (1+2bc\lambda)\partial_\lambda q][2cq - 2bcq'' - \frac{3}{2}q^2]dx \end{aligned}$$

Computing all integrals, dividing by $\|\psi_0\|^2 = \|B^{1/2}q\|^2 = \int_{-\infty}^{\infty}[q^2 + b(q')^2]dx$ to account for the fact that $B^{1/2}q$ was not a unit vector gives the following function

$$H(\lambda,\mu) = \frac{\mu^4(bc\lambda(bc\lambda - 4) - 6) - 10c\lambda\mu^2(bc\lambda + 2) - 15c^2\lambda^2}{4c^2\lambda^2\mu^2\left(b\mu^2 + 5\right)\left(a - bc^2\right)}$$

Use $\mu = \frac{1-c^2}{a-bc^2}, \quad \lambda = \frac{c}{a-bc^2}$ to get

$$H(a,b,c) = \frac{a^2\left(c^4 + 8c^2 + 6\right) - 2ab\left(4c^4 + 7c^2 + 4\right)c^2 + b^2\left(6c^4 + 8c^2 + 1\right)c^4}{4c^4\left(c^2 - 1\right)\left(bc^2 - a\right)\left(b\left(6c^2 - 1\right) - 5a\right)}$$

Notice that the last expression depends only on the variables $z = c^2$ and $d = \frac{a}{b}$, which gives

$$H(z,d) = \frac{d^2z^2 + 8d^2z + 6d^2 - 8dz^3 - 14dz^2 - 8dz + 6z^4 + 8z^3 + z^2}{4(z-1)z^2(z-d)(-5d + 6z - 1)},$$

where $H(c^2, \frac{a}{b}) = H(a,b,c)$. Since the denominator of the last function is always negative for the cases in consideration ($c^2 < 1, c^2 < \frac{a}{b}$), thus the sign of H is governed by its numerator. Define

$$F(z,d) = d^2z^2 + 8d^2z + 6d^2 - 8dz^3 - 14dz^2 - 8dz + 6z^4 + 8z^3 + z^2.$$

If $F(z,d) < 0$ we have $H \geq 0$, hence instability, in the opposite case we can compute the threshold speed c^*. Assuming $H(z,d) < 0$, using Theorem 2 we need to compute z^* such that

$$4z^*(-H(z^*,d)) = 1.$$

Thus the equation for z^* is given by

$$\frac{d^2z^2 + 8d^2z + 6d^2 - 8dz^3 - 14dz^2 - 8dz + 6z^4 + 8z^3 + z^2}{(z-1)z(z-d)(-5d + 6z - 1)} = -1$$

Solving this equation for $d = d(z)$ gives two solutions

$$f(z) = \frac{19z^3 + 2z^2 + 9z - (z-1)z\sqrt{73z^2 + 54z + 33}}{2\left(6z^2 + 3z + 6\right)}$$

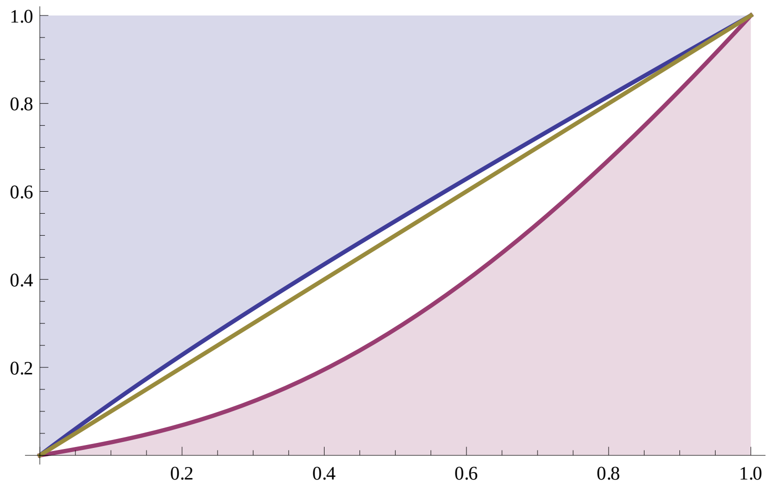

FIGURE 2. Stability index regions

and

$$f_1(z) = \frac{19z^3 + 2z^2 + 9z + (z-1)z\sqrt{73z^2 + 54z + 33}}{2\left(6z^2 + 3z + 6\right)}.$$

Since $c^2 < \frac{a}{b}$ we claim that f_1 (the solution bellow the line $z = d$) is not a possibility, see Fig. 2. Thus the threshold speed is given by $c^* = \sqrt{z^*}$, where $z^* = f^{-1}(d)$ and the waves are stable in the region above the curve $d = f(z)$. The wave speeds between $d = f(z)$ and $d = z$ are linearly unstable, see also Fig. 1.

3. Traveling Standing Waves for the Klein-Gordon Equation

In this section, we setup the linearized problem for (1). We show that the problem also reduces to the appropriate form needed to use the theory in [15], [16]. We prove that the required spectral conditions hold for the operators at hand and compute the values of the parameters that produce stable waves for $p < 5$. For $p \geq 5$ all waves are unstable.

3.1. **Setting up the linearized problem.** To this end, we take the perturbation in the form

$$u = e^{i\omega t} e^{iq(x-ct)}(\varphi(x-ct) + v(t, x-ct)).$$

Plugging this ansatz in (1) and ignoring all quadratic and higher order terms yields the following *linear* equation for v

$$v_{tt} - 2cv_{tx} - 2i(\omega + cq)v_t + (-\mu^2\partial_x^2 + \nu - \varphi^{p-1})v - (p-1)\varphi^{p-1}\Re v = 0.$$

Splitting the real and imaginary parts $v = w + iz$ allows us to rewrite the linearized problem as the following system

$$\left\{\begin{array}{l} w_{tt} - 2cw_{tx} + 2(\omega + cq)z_t + L_+ w = 0 \\ z_{tt} - 2cz_{tx} - 2(\omega + cq)w_t + L_- z = 0, \end{array}\right. \tag{26}$$

where

$$\begin{aligned} L_+ &= -\mu\partial_x^2 + \nu - p\varphi^{p-1} \\ L_- &= -\mu\partial_x^2 + \nu - \varphi^{p-1} \end{aligned}$$

With these assignments, the linearized problem can be written as

$$\begin{pmatrix} w \\ z \end{pmatrix}_{tt} + 2\begin{pmatrix} -c\partial_x & (\omega+qc) \\ -(\omega+qc) & -c\partial_x \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix}_t + \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} = 0 \tag{27}$$

Thus, we may rewrite it further as

$$\mathbf{v}_{tt} + \mathcal{J}\mathbf{v}_t + \mathcal{H}\mathbf{v} = 0, \tag{28}$$

where

$$\mathbf{v} = \begin{pmatrix} w \\ z \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}, \quad \mathcal{J} = 2\begin{pmatrix} -c\partial_x & (\omega+qc) \\ -(\omega+qc) & -c\partial_x \end{pmatrix}.$$

Note that $\mathcal{H}^* = \mathcal{H}$, while $\mathcal{J}^* = -\mathcal{J}$. If we consider the eigenvalue problem associated with (28), that is $\mathbf{v} = e^{\lambda t}\psi$, we arrive at

$$\lambda^2\psi + \lambda\mathcal{J}\psi + \mathcal{H}\psi = 0 \tag{29}$$

We change variables again in (29), namely set $\tilde{\psi} : \psi = e^{-\frac{c\lambda}{\mu}x}\tilde{\psi}$. Note that we need to show that if (29) has a solution in L^2, then indeed $\tilde{\psi}$ is also a solution in L^2 (that is it has the required decay as $x \to \pm\infty$), see Lemma 4. Sidestepping this issue however allows us to reformulate (29) in a form, where the first order derivative term is missing from the operator $\mathcal{J}$. More precisely, we get the following system for $\tilde{\psi}$:

$$\frac{\lambda^2}{\mu}\tilde{\psi} + 2\lambda\begin{pmatrix} 0 & (\omega+qc) \\ -(\omega+qc) & 0 \end{pmatrix}\tilde{\psi} + \mathcal{H}\tilde{\psi} = 0 \tag{30}$$

Multiplying by μ yields an equation for $\tilde{\psi}$, which we will study

$$\lambda^2\tilde{\psi} + 2\lambda\tilde{\mathcal{J}}\tilde{\psi} + \mu\mathcal{H}\tilde{\psi} = 0, \tag{31}$$

where

$$\tilde{\mathcal{J}} := \mu\begin{pmatrix} 0 & (\omega+qc) \\ -(\omega+qc) & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}.$$

This is exactly the form needed to apply the abstract Theorem 3, where we have $\omega = 1$.

3.2. **Proof of the main result for the Klein-Gordon equation.** Let us discuss now the spectral information available for the self-adjoint operator $\mathcal{H}$. The operator $\mathcal{H}$ has a simple negative eigenvalue. Indeed, we have shown that L_+ has a single negative eigenvalue of multiplicity one with an eigenfunction ϕ, while $L_- \geq 0$. Next, the (normalized) elements of the kernel of $\mathcal{H}$ are

$$\psi_0 = \begin{pmatrix} 0 \\ \varphi\|\varphi\|^{-1} \end{pmatrix}, \psi_1 = \begin{pmatrix} \varphi'\|\varphi'\|^{-1} \\ 0 \end{pmatrix}.$$

Further, $\left\langle \tilde{\mathcal{J}}\psi_0, \psi_1 \right\rangle = \omega\|\varphi\|^{-1}\|\varphi'\|^{-1} \ \langle\varphi, \varphi'\rangle = 0$. Clearly, the subspaces

$$X^- = \left\{ \begin{pmatrix} 0 \\ f \end{pmatrix}, f \in L^2 \right\}, X^+ = \left\{ \begin{pmatrix} g \\ 0 \end{pmatrix}, g \in L^2 \right\}$$

provide the split required in Theorem 3. Indeed, $\tilde{\mathcal{J}} : X^{\pm} \to X^{\mp}$, $\psi_0 \in X^-$, $\psi_1 \in X^+$. In addition, $\begin{pmatrix} \phi \\ 0 \end{pmatrix} \in X^+$, recall that $\begin{pmatrix} \phi \\ 0 \end{pmatrix}$ is the eigenfunction of $\mathcal{H}$ corresponding to the negative eigenvalue. Thus, according to Theorem 3, the stability of the quadratic pencil (31) holds if and only if

$$\left\langle \mathcal{H}^{-1}\tilde{\mathcal{J}}\psi_0, \tilde{\mathcal{J}}\psi_0 \right\rangle < 0 \text{ and } 1 \geq \frac{\sqrt{\mu}}{2\sqrt{-\left\langle \mathcal{H}^{-1}\tilde{\mathcal{J}}\psi_0, \tilde{\mathcal{J}}\psi_0 \right\rangle}}$$

We compute that

$$\left\langle \mathcal{H}^{-1}\tilde{\mathcal{J}}\psi_0, \tilde{\mathcal{J}}\psi_0 \right\rangle = \omega^2\|\varphi\|^{-2}\left\langle \mathcal{H}^{-1}\begin{pmatrix} \varphi \\ 0 \end{pmatrix}, \begin{pmatrix} \varphi \\ 0 \end{pmatrix} \right\rangle = \omega^2\|\varphi\|^{-2}\left\langle L_+^{-1}\varphi, \varphi \right\rangle.$$

But differentiating (4) with respect to ν yields $L_+[\frac{\partial\varphi}{\partial\nu}] + \varphi = 0$, whence $L_+^{-1}\varphi = -\frac{\partial\varphi}{\partial\nu}$. Hence,

$$\left\langle \mathcal{H}^{-1}\tilde{\mathcal{J}}\psi_0, \tilde{\mathcal{J}}\psi_0 \right\rangle = -\frac{\omega^2\|\varphi\|^{-2}}{2}\partial_\nu[\|\varphi\|^2].$$

Now, given that the solution of (4) is in the form

$$\varphi_{\mu,\nu}(x) = \nu^{\frac{1}{p-1}}\varphi_0(\frac{\sqrt{\nu}}{\sqrt{\mu}}x)$$

It follows that

$$\left\langle \mathcal{H}^{-1}\tilde{\mathcal{J}}\psi_0, \tilde{\mathcal{J}}\psi_0 \right\rangle = \frac{\omega^2}{\nu}\left(\frac{1}{4} - \frac{1}{p-1}\right).$$

Thus, for stability, we need $p < 5$ and in addition, solving the relation $1 \geq \frac{\sqrt{\mu}}{2\sqrt{-\left\langle \mathcal{H}^{-1}\tilde{\mathcal{J}}\psi_0, \tilde{\mathcal{J}}\psi_0 \right\rangle}}$ yields

$$1 < p \leq 1 + \frac{4\omega^2}{1-c^2},\ \omega^2 + c^2 < 1.$$

This is the characterization of the stability of the waves.

Appendix A. Decay of the unstable solution of (29)

Lemma 4. *Let ψ is the L^2 solution of (29). Then, $\tilde{\psi}(x) = e^{\frac{c\lambda}{\mu}x}\psi \in L^2(\mathbf{R}^1)$. In fact, there is C, so that*

$$|\tilde{\psi}(x)| \leq Ce^{-\frac{\lambda}{\mu}|x|}.$$

Proof. Normalize $\psi : \|\psi\|_{L^2} = 1$. Starting from (29), it is easy to obtain estimates for $\|\psi''\|_{L^2}$ in terms of $\|\psi\|_{H^1}$ in the form

$$\|\psi''\|_{L^2} \leq C(\|\psi'\|_{L^2} + \|\psi\|_{L^2}) \leq \frac{1}{2}\|\psi''\|_{L^2} + C_1\|\psi\|_{L^2},$$

whence

$$\|\psi''\|_{L^2} \leq 2C_1.$$

It follows that ψ is bounded, with $\|\psi\|_{L^\infty}$ depending on the parameters. Clearly $|\tilde{\psi}(x)| \leq Ce^{\beta|x|}$, where $\beta \leq \frac{c\lambda}{\mu}$. Let us setup the integral equation for $\tilde{\psi}$, based on (30). Diagonalizing the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ leads to

$$\begin{pmatrix} 0 & \frac{\omega}{\mu} \\ -\frac{\omega}{\mu} & 0 \end{pmatrix} = S \begin{pmatrix} -i\frac{\omega}{\mu} & 0 \\ 0 & i\frac{\omega}{\mu} \end{pmatrix} S^{-1}, \quad S = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}.$$

Thus, we can rewrite (30) as follows

(32)
$$S\left(\frac{\lambda^2}{\mu} Id + 2\lambda \begin{pmatrix} -i\frac{\omega}{\mu} & 0 \\ 0 & i\frac{\omega}{\mu} \end{pmatrix} + \begin{pmatrix} -\mu\partial_x^2 + \nu & 0 \\ 0 & \mu\partial_x^2 + \nu \end{pmatrix}\right) S^{-1}\tilde{\psi} = \begin{pmatrix} p\varphi^{p-1} & 0 \\ 0 & \varphi^{p-1} \end{pmatrix} \tilde{\psi},$$

Introducing the new variable $Z = S^{-1}\tilde{\psi}$ and the fast decaying potential

$$\mathcal{V} := S^{-1} \begin{pmatrix} p\varphi^{p-1} & 0 \\ 0 & \varphi^{p-1} \end{pmatrix} S,$$

we rewrite (32) as

$$\left(\frac{\lambda^2}{\mu} Id + 2\lambda \begin{pmatrix} -i\frac{\omega}{\mu} & 0 \\ 0 & i\frac{\omega}{\mu} \end{pmatrix} + \begin{pmatrix} -\mu\partial_x^2 + \nu & 0 \\ 0 & \mu\partial_x^2 + \nu \end{pmatrix}\right) Z = \mathcal{V}Z. \tag{33}$$

Clearly, we then need to invert the constant coefficient differential operators (after dividing by μ)

$$-\partial_x^2 + \left(\frac{\lambda^2 + \nu\mu}{\mu^2} \pm 2\lambda i \frac{\omega}{\mu^2}\right)$$

Let $k_\pm$ be so that

$$k_\pm^2 = \frac{\lambda^2 + \nu\mu}{\mu^2} \pm 2\lambda i \frac{\omega}{\mu^2}.$$

Setting $k_\pm = a \pm ib, a > 0$, observe that

$$a^2 \geq a^2 - b^2 = \Re k_+^2 = \frac{\lambda^2 + \nu\mu}{\mu^2} > \frac{\lambda^2}{\mu^2},$$

hence $a > \frac{\lambda}{\mu} > 0$. It follows that one can setup an integral equation for Z

$$Z(x) = c\int_{-\infty}^{\infty} \begin{pmatrix} e^{-k_-|x-y|} & 0 \\ 0 & e^{-k_+|x-y|} \end{pmatrix} \mathcal{V}(y)Z(y)dy.$$

Estimate by taking absolute values. Note that $\mathcal{V}$ has exponential decay, say $|\mathcal{V}(y)| \leq e^{-|\alpha|y}$ for some $\alpha > 0$. We obtain

$$|Z(x)| \leq c\int_{-\infty}^{\infty} e^{-a|x-y|}e^{-\alpha|y|}|Z(y)|dy. \tag{34}$$

Thus, starting with out *a priori* assumption $|Z(y)| \leq Ce^{\beta|y|}$, and noting that $a > \frac{\lambda}{\mu} > \frac{c\lambda}{\mu} \geq \beta$, we obtain from (34), say for $x > 0$,

$$\begin{aligned} |Z(x)| &\leq C(\int_x^\infty e^{-a(y-x)} e^{-\alpha y} e^{\beta y} dy + \int_0^x e^{-a(x-y)} e^{-\alpha y} e^{\beta y} dy + \int_{-\infty}^0 e^{-a(x-y)} e^{\alpha y} e^{-\beta y} dy) \leq \\ &\leq C(e^{(\beta-\alpha)x} + e^{-ax}). \end{aligned}$$

Similar argument in the case $x < 0$ implies that for all x,

$$|Z(x)| \leq C(e^{(\beta-\alpha)|x|} + e^{-a|x|}).$$

It follows that the estimate $|Z(x)| \leq Ce^{\beta|x|}$ can be upgraded to $Ce^{(\beta-\alpha)|x|}$ (recall $\alpha > 0$), unless $\beta \leq -a$. By iterating this argument, if needed, it follows that

$$|Z(x)| \leq Ce^{-a|x|} \leq Ce^{-\frac{\lambda}{\mu}|x|},$$

which implies that $\tilde{\psi} = SZ$ satisfies the same estimate.

□

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