DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KANSAS
MATH 765 - Fall 2007 - Final Exam

Your Name: ________________________________

On this exam, you may NOT use books and/or notes.

1	(50)	________
2	(50)	________
3	(50)	________
4	(50)	________
5	(50)	________
6	(50)	________
7	(50)	________
8	(50)	________
9	(50)	________
10	(50)	________

(1) Prove

$$\sum_{k=1}^{n}(2k-1)^2 = \frac{n(4n^2-1)}{3}$$

(2) Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers, so that $x_n \to x, y_n \to x$, where x is an extended real number (i.e. x could be $\pm\infty$). Show that if $x_n \leq w_n \leq y_n$, then $w_n \to x$. In your proof, rely exclusively on the definition of limits.

(3) Suppose that $x_0 \in \mathbf{R}$ and $x_{n+1} = (1 + x_n)/2$. Prove that $x_n \to 1$.

(4) Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function and
$$\lim_{x \to +\infty} f(x) = \infty = \lim_{x \to -\infty} f(x)$$
Prove that f achieves its minimum on $\mathbf{R}$, that is there exists $x_m \in \mathbf{R}$, so that
$$f(x_m) = \inf_{x \in \mathbf{R}} f(x) < \infty.$$

(5) Let $f : (a, b) \to \mathbf{R}$ be differentiable function on the non-empty interval (a, b), such that f' is bounded on (a, b). Prove that f is uniformly continuous on (a, b).

(6) Suppose $f : [a, b] \to \mathbf{R}$ and f is Riemann integrable. Prove that

- If f is continuous at $x_0 \in (a, b)$ and $f(x_0) \neq 0$, then

$$\int_a^b |f(x)|dx > 0.$$

- if f is in addition continuous on $[a, b]$, then $\int_a^b |f(x)|dx = 0$ if and only if $f(x) = 0$ for all $x \in [a, b]$.

(7) Suppose that $a_k \to 0$ and a_k is decreasing. Prove that $\sum_{k=0}^{\infty} a_k \sin(kx)$ converges uniformly on any closed interval $[a, b] \subset (0, 2\pi)$. What goes wrong in $[0, 2\pi]$.
Hint: Note that this does NOT follow from the M test. Instead, try to use the summation by parts formula as we did in class.
Bonus: 20 points Counterexamples anyone? That is, try to produce a function in the form $f(x) = \sum_k a_k \sin(kx)$, so that the convergence is not uniform.

(8) Let f, g be continuous on a closed bounded interval $[a, b]$, so that $|g(x)| > 0$ for $x \in [a, b]$. Suppose $f_n \to f$, $g_n \to g$ uniformly on $[a, b]$. Prove that $f_n/g_n \to f/g$ uniformly on $[a, b]$.
Hint: Prove first that there exists $b > 0$, so that $|g(x)| \geq b$.

(9) Find a closed form expression (i.e. an explicit formula) for the sum

$$\sum_{k=0}^{\infty}(k+1)x^k.$$

(10) Let f be Riemann integrable on $[0,1]$ and (right) continuous at 0. Let $\alpha > 0$. Compute

$$\lim_{n\to\infty} n^{\alpha} \int_0^{n^{-\alpha}} f(x)dx.$$

Justify your computations.