DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KANSAS
Final Exam
MATH 765 Fall 2010

Your Name: ______________________________

1	(50)	________
2	(50)	________
3	(50)	________
4	(75)	________
5	(75)	________
6	(75)	________
7	(75)	________
8	(75)	________
Total	(500)	________

(1) (50 points) Prove the following inequalities for all reals:

$$1 + x \leq e^x \leq 1 + xe^x.$$

(2) (50 points)
Show that if the radius of convergence of the power series $\sum_{k=0}^{\infty} a_k x^k$ is finite, then show that the radius of convergence of the series $\sum_k a_k x^{k^2}$ is exactly one.

(3) (50 points) Let $f, g : (a, b) \to \mathbf{R}^1$ be a Riemann integrable function. Show that their product is also Riemann integrable on (a, b).

(4) (75 points)
Suppose that $f : (a, b) \to \mathbf{R}^1$ and f is differentiable at $x_0 \in (a, b)$. Let $\alpha, \beta > 0$. Show that

$$\lim_{n\to\infty} n \left[f\left(x_0 + \frac{\alpha}{n}\right) - f\left(x_0 - \frac{\beta}{n}\right) \right] = (\alpha + \beta) f'(x_0).$$

Bonus: 10 points. Show by an example that for some α, β, f, the limit above may exist, but the function f is not differentiable at x_0.

(5) (75 points)
Let $f : (0,1) \to \mathbf{R}^1$ be a differentiable function, such that $|f'(x)| \leq M$ for all $x \in (0,1)$. Show that

$$\left| \int_0^1 f(x)dx - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \right| \leq \frac{M}{2n^2}.$$

Hint: Write $\int_0^1 f(x)dx = \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} f(x)dx$

(6) (75 points)
Let $f_n : [a, b] \to \mathbf{R}^1$ be a sequence of continuous functions, which converges uniformly to f. Show that the sequence $\{f_n\}_n$ is uniformly bounded.

That is, show that there exists M, so that for all $n \in \mathbf{N}$ and all $x \in [a, b]$, there is the inequality

$$|f_n(x)| \leq M.$$

(7) (75 points)
Let $f_n : [a, b] \to \mathbf{R}^1$ be a sequence of continuous functions, which converges uniformly to f. Show that if $x_n \in [a, b] : x_n \to x_0$, one has $f_n(x_n) \to f(x_0)$.

(8) (75 points)
Prove that if the terms of the series $\sum_k a_k$ are non-negative and decrease monotonically to zero, then $\sum_k a_k$ converges if and only if the series $\sum_j (2j+1)a_{j^2}$ converges.