PERIODIC TRAVELLING WAVES OF THE SHORT PULSE EQUATION: EXISTENCE AND STABILITY

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Abstract. We construct various periodic travelling wave solutions of the Ostrovsky/Hunter-Saxton/short pulse equation and its KdV regularized version.

For the regularized short pulse model with small Coriolis parameter, we describe a family of periodic travelling waves which are a perturbation of appropriate KdV solitary waves. We show that these waves are spectrally stable.

For the short pulse model, we construct a family of travelling peakons with corner crests. We show that the peakons are spectrally stable as well.

1. Introduction

It is well-known that the (generalized) Korteweg-De Vries equation

$$u_t + \beta u_{xxx} + (f(u))_x = 0, \tag{1}$$

can be used in the modelling and understanding the dynamics of large scales in the atmosphere and in the oceans. There is a substantial amount of research into various aspects of this model - well-posedness, propagation of singularities, asymptotic behavior near solitary waves etc. We will not even attempt to review these here, as this is outside the scope of this paper. Instead, we discuss a related model, which takes into account the effect of a rotation force, if such is applied to the fluid. To be sure, the effect of the rotation of the Earth (which induces the so-called Coriolis force) is often a negligible factor. As such, it is not taken into account in the derivation of various water wave models, such as (1). However, adding this feature in the model results in the following dispersive model

$$(u_t + \beta u_{xxx} + (f(u))_x)_x + \epsilon u = 0, -T \leq x \leq T. \tag{2}$$

A few things are worth noting here. First, we adopt the notation ϵ in order to emphasize the fact that usually $0 < |\epsilon| << 1$ is a small parameter and also, at least mathematically, we can allow ϵ to be both positive or negative[1]. Another feature of the model (2) is that we consider it in a finite, spatially symmetric interval, with appropriate boundary conditions, to be specified later on.

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[1]Although in a physically relevant models, ϵ is a small negative number

This is a model which has gained popularity lately.[2] We refer to it as the regularized short pulse equation (RSPE). An interesting property of (2) is that since the left hand side is an exact derivative of a periodic function, the solution u must be mean value zero for all times t. Because of this feature, at least in the whole line case, things get subtle at this point. In particular, even though one may still conclude from (4) that $\int_{-\infty}^{\infty} u(t,x)dx = 0$, this is not enough to properly define $\partial_x^{-1} u$. Regardless of these issues, in [14, 15] (under some assumptions on the coefficient β, the function f and ϵ), Levandosky and Liu have succeeded in constructing travelling wave solutions of (2) on the whole line by employing variational methods. They have also studied the stability of such solutions by relying on the Grillakis-Shatah-Strauss theory, although their results do not give a full stability picture of all solitary waves constructed in their paper. Further work on the stability of these travelling waves was done by Liu, [16] and Liu-Ohta, [17].

In [5], the authors have considered the same problem for small values of ϵ. They have constructed solutions of (4) via singular perturbation theory. In [6], in collaboration with Sandstede, they have extended their previous results to include the existence of multipulse solutions. To the best of our knowledge, the problem for the stability of both types of waves remains open.

An interesting special model occurs, when the KdV regularization is absent[3], in other words, $\beta = 0$. This is referred to in the literature, depending on the form of the nonlinearity f, as the reduced Ostrovsky/Ostrovsky-Hunter/short pulse model. Namely, we scale for convenience $\epsilon = 1$, which leads us to

$$(u_t + (f(u))_x)_x = u. \tag{3}$$

Such a model represents an independent interest, not necessarily related to the (generalized) KdV equation. In fact, (3), with various form of the nonlinearity has rich history. The first model of this family was introduced by Ostrovsky, [21] in the late 70's. In the early 90's, Vakhnenko, [28] came up with an alternative derivation, while Hunter, [10] proposed some numerical simulations. The well-posedness questions were investigated by Boyd, [2]; Schaefer and Wayne, [25]; Stefanov-Shen-Kevrekidis, [26]. Liu, Pelinovsky and Sakovich, [18, 19] have studied wave breaking, which was later supplemented by the global regularity results of Grimshaw-Pelinovsky, [8]. There are numerous works on explicit travelling wave solutions of these models, [7, 20, 23, 24, 27, 29, 30]. One should note that some of this solutions are not classical solutions, but rather a multi-valued ones, [24]. Several authors have also explored the integrability of the Ostrovsky equation, [24, 30], in particular they have managed to construct the traveling waves by means of the inverse scattering transform. Finally, there are several attempts at stability, like [22], [23] . These mostly consists of either direct numerical simulations of the full equation or a numerical spectral computations, performed on the linearized operator.

One of the main goals of this work is to provide almost explicit solutions of (3) in the case of quadratic nonlinearity[4] and in addition, to provide definite results about their stability.

[2]One particularly interesting aspect of it is the connection to the two dimensional KP equations - indeed, (2) provides special one dimensional solutions of KP, which may be used as a reference in the study of the dynamics of KP.

[3]One should note that the presence of $\beta \neq 0$, provides a regularizing effect on the evolution, [7]

[4]although, one can, expand our approach beyond the case $f(u) = u^2$

In fact, we show the existence of certain periodic waves, which turn out to have a crest singularity at the ends of the interval, that is peakon solutions. In addition, we show that such solutions are spectrally stable.

Since we will be interested in travelling wave solutions of (2) and their stability properties, we should give a more precise definition and record the corresponding elliptic equations. More specifically, we consider solutions of (2) in the form $\varphi(x-ct)$. They satisfy

$$-c\varphi'' + \beta\varphi'''' + (f(\varphi))'' + \epsilon\varphi = 0, -T \le x \le T. \tag{4}$$

In the periodic context *and if one assume that φ is sufficiently smooth*, this formulation immediately implies that $\int_{-T}^{T} \varphi(x)dx = 0$.

1.1. Some notations.

We will often work in the subspace $L^2_0 = \{f \in L^2[-T,T] : \int_{-T}^{T} f(x)dx = 0\}$, which can be realized as the space of all Fourier series with the restriction $a_0 = 0$. Namely, $f \in L^2_0$ if and only if $f(x) = \frac{1}{\sqrt{2T}} \sum_{k=-\infty, k\neq 0}^{\infty} a_k e^{\pi i k x/T}$, with a norm given by

$$\|f\|_{L^2_0} = \left(\sum_{k=-\infty, k\neq 0}^{\infty} |a_k|^2 \right)^{1/2},$$

where $a_k = \frac{1}{\sqrt{2T}} \int_{-T}^{T} f(x) e^{-\pi i k x/T} dx$. The Sobolev spaces with mean zero functions are defined $H^s_0[-T,T] = L^2_0 \cap H^s[-T,T]$. In addition, one can consider the action of the operator ∂_x^{-1} on L^2_0 and the corresponding Sobolev spaces H^s_0, defined in a standard way by the formula

$$\partial_x^{-1} f = T \sum_{k=-\infty, k\neq 0}^{\infty} \frac{a_k}{\pi i k} e^{\pi i k x/T}.$$

More generally, we introduce the following operators, which will be useful in the sequel. On the Sobolev space with zero mean functions, H^s_0 (where s is a real parameter), let

$$|\partial_x|^s f(x) := T^{-s} \sum_{k=-\infty, k\neq 0}^{\infty} \pi^s |k|^s a_k e^{\pi i k x/T} \tag{5}$$

In addition, introduce the Hilbert transform operator $H : H\partial_x = |\partial_x|$ or $\partial_x = -H|\partial_x|$. In other words,

$$Hf(x) = i \sum_{k=-\infty, k\neq 0}^{\infty} sgn(k) a_k e^{\pi i k x/T} \tag{6}$$

It is easy to see that H is a skew-symmetric operator, that is $H^* = -H$. In addition, H maps real-valued functions into real-valued functions.

1.2. Organization of the paper and a synopsis of the main results.

The main goal of this work is to provide some new periodic travelling wave solutions of the short pulse equation (3) and its regularized version, (2). For the regularized SPE (2), we can construct a family of waves, for all small ϵ, whenever the underlying KdV equation (1) has such waves and they satisfy appropriate non-degeneracy requirements. This is done in Section 2 below via the Lyapunov-Schmidt reduction argument. We apply these results, for example[5], to the cnoidal solutions of KdV constructed in Section 2.1. In Section 3, we establish the spectral stability of the solutions constructed perturbatively in Section 2. Our approach relies on the "JL method" of Kevrekidis-Kapitula-Kevrekidis, [12, 13], see also the related work of Chugunova-Pelinovsky, [4]. In Section 4, we take on the short pulse equation (3), in the case $f(u) = -u^2$. We construct peakon type solutions. These are continuous functions in an interval $[-T, T]$, but their derivatives develop a jump discontinuity at the points $\pm T$. That is, they have a corner crest. In Section 5, we show that these periodic travelling peakons are spectrally stable.

2. Construction of the periodic travelling waves for the regularized short pulse equation with small Coriolis parameter

In this section, we first state and prove a general theorem for the existence of periodic travelling waves for RSPE, for small $\epsilon << 1$. After that, we give concrete and explicit of travelling wave solutions of the KdV equation, which satisfy the assumption in Theorem 1 below.

Before we formulate the main result, let us rewrite the equation (4) in a more suitable form for our analysis. For $\epsilon = 0$, the equation (4) reduces to $-\partial_x^2[-\beta\varphi'' + c\varphi - f(\varphi)] = 0$. After two integrations in the x variable, this reduces[6] to

$$-\beta\varphi'' + c\varphi - f(\varphi) = h, \qquad -T \leq x \leq T. \tag{7}$$

For the general case, under the condition $\int_{-T}^{T} \varphi(x)dx = 0$, we are looking for a solution in the form

$$-\beta\varphi_\epsilon'' + c\varphi_\epsilon - f(\varphi_\epsilon) - \epsilon\partial_x^{-2}\varphi_\epsilon = h + \alpha(\epsilon), \varphi_\epsilon \in L^2_0[-T, T] \tag{8}$$

for some $\alpha = \alpha(\epsilon)$. Indeed, any solution of (8) will be a solution to (4), just apply ∂_x^2 on both sides of (8). So, our goal is to produce a solution to (8) for $\epsilon << 1$.

Our next result states that one can perturbatively produce a solution of (8) starting from a solution of (7).

Theorem 1. *Let the non-linearity f be so that, $f \in C^2(\mathbf{R}^1), f(0) = 0$ and the following holds*

1. *There is an even, mean zero and smooth solution $\varphi_0 \in L^2$ of (7).*
2. *The linearized operator $L_+ := -\beta\partial_x^2 + c - f'(\varphi_0)$ has kernel spanned by odd functions. That is $Ker[L_+] \subset L^2_{odd}$.*
3. *$\langle L_+^{-1}[1], 1\rangle \neq 0$.*

[5]We would like to point out on the other hand, that our results are pretty flexible in that they should be applicable to a wide range of examples, satisfying a set of assumptions.

[6]Note that in principle we obtain linear terms $h + mx$ on the right hand side. But, the cases with $m \neq 0$ are not permissible, due to the periodicity requirements in $[-T, T]$, hence $m = 0$.

Then, there exists $\epsilon_0 > 0$, a function $\alpha(\epsilon)$ and a function $\varphi_\epsilon \in H^2_{0,even}$, which is a solution to the equation (8). *In addition,*

$$\varphi_\epsilon = \varphi_0 + \epsilon L_+^{-1}\left[\partial_x^{-2}\varphi_0 - \frac{\langle \partial_x^{-2}\varphi_0, L_+^{-1}[1]\rangle}{\langle 1, L_+^{-1}[1]\rangle}\right] + O(\epsilon^2) \tag{9}$$

$$\alpha(\epsilon) = -\epsilon\frac{\langle \partial_x^{-2}\varphi_0, L_+^{-1}[1]\rangle}{\langle 1, L_+^{-1}[1]\rangle} + O(\epsilon^2). \tag{10}$$

Proof. Define $\psi_0 = \frac{L_+^{-1}[1]}{\|L_+^{-1}[1]\|}$. This is a well-defined element of $H^2_{even}[-T, T]$ by the Fredholm alternative, since $1 \perp Ker[L_+]$, which follows from $Ker[L_+] \subset L^2_{odd}$.

For $\psi \in H^2_{0,even}$, introduce $F : \mathbf{R}^1 \times \mathbf{R}^1 \times H^2_{0,even} \to L^2_{even}$ defined via

$$F(\epsilon, \alpha; \psi) = -\beta(\varphi_0 + \psi)'' + c(\varphi_0 + \psi) - \epsilon\partial_x^{-2}(\varphi_0 + \psi) - f(\varphi_0 + \psi) - h - \alpha.$$

Note that the condition $\psi \in L^2_0$ allows us to take the operator ∂_x^{-2} in the definition of F.

Let $P_{\{\psi_0\}^\perp}$ be the orthogonal projection on the subspace $\{\psi_0\}^\perp = \{\psi : \langle\psi, \psi_0\rangle = 0\}$. That is

$$P_{\{\psi_0\}^\perp} f = f - \langle f, \psi_0\rangle\psi_0.$$

We prove our theorem via the Lyapunov-Schmidt procedure. First, $F(0, 0; 0) = 0$.

Next, we show that there is $\psi = \psi(\epsilon, \alpha) \in H^2_{0,even}$, so that the equation

$$P_{\{\psi_0\}^\perp} F(\epsilon, \alpha; \psi) = 0, \tag{11}$$

holds true for (ϵ, α) in some neighborhood of $(0, 0)$. As an application of the implicit function theorem for (11), in the spaces indicated above, we will need to show that

$$D_\psi(0, 0; 0) : H^2_{0,even} \to Y$$

is an invertible operator, where $Y = P_{\{\psi_0\}^\perp}[L^2_{0,even}] \subset L^2_{0,even}$. We have

$$D_\psi(0, 0; 0)\tilde{\psi} = P_{\{\psi_0\}^\perp}[-\beta\tilde{\psi}'' + c\tilde{\psi} - f'(\varphi_0)\tilde{\psi}] = P_{\{\psi_0\}^\perp} L_+[\tilde{\psi}].$$

Let $g \in Y$ be an arbitrary element and consider $D_\psi(0, 0; 0)\tilde{\psi} = g$. That is

$$P_{\{\psi_0\}^\perp} L_+[\tilde{\psi}] = g.$$

Since $Ker[L_+]$ is spanned by odd functions, L_+ is invertible on L^2_{even} and hence $\tilde{\psi} = L_+^{-1}[g] \in H^2_{even}$ is well defined. It remains to check that $\tilde{\psi}$ has mean value zero. We have by the self-adjointness of L_+ on L^2_{even}

$$\langle\tilde{\psi}, 1\rangle = \langle L_+^{-1}[g], 1\rangle = \langle g, L_+^{-1}[1]\rangle = \|L_+^{-1}[1]\|\langle g, \psi_0\rangle = 0,$$

since $g \in Y = P_{\{\psi_0\}^\perp}[L^2_{0,even}]$. Thus, $\tilde{\psi}$ has mean value zero and the first step of the Lyapunov-Schmidt procedure is justified. That is, there is a function $\psi = \psi(\epsilon, \alpha)$.

For the second step, we need to resolve the remaining equation, namely

$$\langle F(\epsilon, \alpha; \psi(\epsilon, \alpha)), \psi_0\rangle = 0. \tag{12}$$

Denoting $f(\epsilon,\alpha)=\langle F(\epsilon,\alpha;\psi(\epsilon,\alpha)),\psi_0\rangle$, we need to check that the implicit function theorem applies, so that there is a solution $\alpha(\epsilon)$. We have

$$\frac{\partial f}{\partial \alpha} = \langle -\beta\left(\frac{\partial \psi}{\partial \alpha}\right)'' + c\frac{\partial \psi}{\partial \alpha} - \epsilon\partial_x^{-2}\left(\frac{\partial \psi}{\partial \alpha}\right) - f'(\varphi_0+a\psi_0+\psi)\frac{\partial \psi}{\partial \alpha} - 1, \psi_0\rangle$$

Evaluating at $\epsilon=0,\alpha=0$ yields

$$\begin{aligned}\frac{\partial f}{\partial \alpha}|_{\epsilon=0,\alpha=0} &= \langle L_+\frac{\partial \psi}{\partial \alpha}|_{\epsilon=0,\alpha=0},\psi_0\rangle - \langle 1,\psi_0\rangle = \\ &= \frac{1}{\|L_+^{-1}[1]\|}\langle\frac{\partial \psi}{\partial \alpha}|_{\epsilon=0,\alpha=0},1\rangle - \langle 1,\psi_0\rangle = -\langle 1,\psi_0\rangle,\end{aligned}$$

where we have used the self-adjointness of L_+, the fact that $\psi(\epsilon,\alpha)\in L^2_0$ (and hence $\frac{\partial\psi}{\partial\alpha}|_{\epsilon=0,\alpha=0}\perp 1$). Finally, by the assumptions in the theorem,

$$\frac{\partial f}{\partial \alpha}|_{\epsilon=0,\alpha=0} = -\frac{1}{\|L_+^{-1}[1]\|}\langle 1, L_+^{-1}[1]\rangle \neq 0,$$

and hence the existence of $\alpha(\epsilon)$ in a small interval $(-\epsilon_0,\epsilon_0)$ is shown.

We now establish the behavior of $\alpha(\epsilon)$ and $\psi_\epsilon := \psi(\epsilon,\lambda(\epsilon))$. We start with the relation

$$P_{\{\psi_0\}^\perp}F(\epsilon,\alpha;\psi(\epsilon,\alpha))=0. \tag{13}$$

Taking a derivative in ϵ and evaluating at $\epsilon=0,\alpha=0$ yields

$$P_{\{\psi_0\}^\perp}[L_+[\frac{\partial \psi}{\partial \epsilon}|_{\epsilon=\alpha=0}]-\partial_x^{-2}\varphi_0]=0.$$

Since $L_+[\frac{\partial\psi}{\partial\epsilon}|_{\epsilon=\alpha=0}]\perp\psi_0$, it follows that

$$L_+[\frac{\partial \psi}{\partial \epsilon}|_{\epsilon=\alpha=0}] = P_{\{\psi_0\}^\perp}[\partial_x^{-2}\varphi_0]$$

The right hand side is even and hence orthogonal to $Ker[L_+]$, whence we can find the unique solution

$$\frac{\partial \psi}{\partial \epsilon}|_{\epsilon=\alpha=0} = L_+^{-1}[P_{\{\psi_0\}^\perp}(\partial_x^{-2}\varphi_0)]. \tag{14}$$

Similarly, taking derivative with respect to α in (13) and evaluating at $\epsilon=\alpha=0$, we obtain

$$P_{\{\psi_0\}^\perp}[L_+[\frac{\partial \psi}{\partial \alpha}|_{\epsilon=\alpha=0}]-1]=0$$

This implies $L_+[\frac{\partial\psi}{\partial\alpha}|_{\epsilon=\alpha=0}]=P_{\{\psi_0\}^\perp}(1)$ and hence

$$\frac{\partial \psi}{\partial \alpha}|_{\epsilon=\alpha=0} = L_+^{-1}[P_{\{\psi_0\}^\perp}(1)], \tag{15}$$

since $P_{\{\psi_0\}^\perp}(1)$ is an even function.

Finally, in order to find $\alpha'(0)$, we take a derivative in ϵ in the equation

$$F(\epsilon,\alpha(\epsilon),\psi(\epsilon,\alpha(\epsilon)))=0.$$

and evaluate at $\epsilon = 0$. This yields

$$L_+[\frac{\partial \psi}{\partial \epsilon}|_{\epsilon=\alpha=0}] + \alpha'(0) L_+[\frac{\partial \psi}{\partial \alpha}|_{\epsilon=\alpha=0}] - \partial_x^{-2}[\varphi_0] = \alpha'(0).$$

Using (14) and (15), this further simplifies to

$$-\langle \partial_x^{-2}\varphi_0, \psi_0\rangle \psi_0 - \alpha'(0)\langle 1, \psi_0\rangle \psi_0 = 0,$$

which results in the formula

$$\alpha'(0) = -\frac{\langle \partial_x^{-2}\varphi_0, \psi_0\rangle}{\langle 1, \psi_0\rangle}. \tag{16}$$

Here again, we have used the condition that $\langle 1, \psi_0\rangle = \langle L_+^{-1}[1], 1\rangle \neq 0$. This allows us to derive the following representation formula for ψ_ϵ,

$$\begin{aligned}
\psi_\epsilon(x) &= \epsilon \frac{\partial \psi}{\partial \epsilon}|_{\epsilon=\alpha=0} + \alpha(\epsilon) \frac{\partial \psi}{\partial \alpha}|_{\epsilon=\alpha=0} + O(\epsilon^2) = \\
&= \epsilon \left[L_+^{-1}[P_{\{\psi_0\}^\perp}(\partial_x^{-2}\varphi_0)] - \frac{\langle \partial_x^{-2}\varphi_0, \psi_0\rangle}{\langle 1, \psi_0\rangle} L_+^{-1}[P_{\{\psi_0\}^\perp}(1)] \right] + O(\epsilon^2) = \\
&= \epsilon L_+^{-1}(P_{\{\psi_0\}^\perp} \left[\partial_x^{-2}\varphi_0 - \frac{\langle \partial_x^{-2}\varphi_0, \psi_0\rangle}{\langle 1, \psi_0\rangle} \right]) + O(\epsilon^2).
\end{aligned}$$

Since

$$\langle \partial_x^{-2}\varphi_0 - \frac{\langle \partial_x^{-2}\varphi_0, \psi_0\rangle}{\langle 1, \psi_0\rangle}, \psi_0\rangle = \langle \partial_x^{-2}\varphi_0, \psi_0\rangle - \frac{\langle \partial_x^{-2}\varphi_0, \psi_0\rangle}{\langle 1, \psi_0\rangle}\langle 1, \psi_0\rangle = 0,$$

we find that $P_{\{\psi_0\}^\perp}\left[\partial_x^{-2}\varphi_0 - \frac{\langle \partial_x^{-2}\varphi_0, \psi_0\rangle}{\langle 1, \psi_0\rangle}\right] = \partial_x^{-2}\varphi_0 - \frac{\langle \partial_x^{-2}\varphi_0, \psi_0\rangle}{\langle 1, \psi_0\rangle}$ and so, we finally have the formula

$$\psi_\epsilon(x) = \epsilon L_+^{-1}\left[\partial_x^{-2}\varphi_0 - \frac{\langle \partial_x^{-2}\varphi_0, \psi_0\rangle}{\langle 1, \psi_0\rangle}\right] + O(\epsilon^2).$$

$\square$

2.1. **Examples of periodic travelling wave solutions of KdV satisfying Theorem 1.** As we have mentioned above, we consider only the KdV case. That is, take $f(z) = z^2$. Integrating once more the equation (8), we get the equation

$$\varphi'^2 = \frac{2}{3\beta}\left(-\varphi^3 + \frac{3}{2}c\varphi^2 + 3h\varphi + 3h_2\right). \tag{17}$$

Let $c > 0$ and $\varphi_1 < \varphi_2 < \varphi_3$ are roots of the polynomial $F(\rho) = -\rho^3 + \frac{3}{2}c\rho^2 + 3h\rho + 3h_2$. Then, we have $F(\rho) = (\rho - \varphi_1)(\rho - \varphi_2)(\varphi_3 - \rho)$ and

$$\left| \begin{array}{l} \varphi_1 + \varphi_2 + \varphi_3 = \frac{3}{2}c \\ \\ \varphi_1\varphi_2 + \varphi_1\varphi_3 + \varphi_2\varphi_3 = -3h \\ \\ \varphi_1\varphi_2\varphi_3 = 3h_2. \end{array} \right. \tag{18}$$

Introducing new variable, s via $\varphi = \varphi_2 + (\varphi_3 - \varphi_2)s^2$, we get

$$s'^2 = \frac{1}{6\beta}(1-s^2)(\kappa'^2 + \kappa^2 s^2)$$

and the solution of equation (17 is given by

$$\varphi(x) = \varphi_2 + (\varphi_3 - \varphi_2)cn^2(\alpha x, \kappa), \tag{19}$$

where

$$\kappa^2 = \frac{2\varphi_3 - 2\varphi_2}{4\varphi_3 + 2\varphi_2 - 3c}, \quad \kappa'^2 = 1 - \kappa^2, \quad \alpha^2 = \frac{4\varphi_3 + 2\varphi_2 - 3c}{12\beta}. \tag{20}$$

For fixed T in a proper interval, we cam determine φ_2 and φ_3 as smooth function of c so that the periodic solution φ given by (19) will have period T because of monotonicity of the period (for more details see [1, 9]). Moreover, for $T > 0$ and $c > 0$ there exists a smooth branch of cnoidal waves with mean zero (see [1]).

In $[-T, T] = [-K(k)/\alpha, K(k)/\alpha]$, we consider the spectral properties of the operator

$$L_+ = -\beta\partial_x^2 + c - 2\varphi,$$

supplied with periodic boundary conditions. By the above formulas, $\varphi_3 - \varphi_2 = 6\alpha^2 k^2$, $2\varphi_2 - c = 4\alpha^2(1 - 2k^2)$. Taking $y = \alpha x$ as an independent variable in L, one obtains $L = \alpha^2\Lambda$ with an operator Λ in $[-K(k), K(k)]$ given by

$$\Lambda = -\partial_x^2 - 4(1 + k^2) + 12k^2 sn^2(y; k).$$

The spectral properties of the operator Λ in $[0, 2K(k)]$ are well known. The first three(simple) eigenvalues and corresponding eigenfunctions of Λ are

$$\begin{aligned}
&\mu_0 = k^2 - 2 - 2\sqrt{1 - k^2 + 4k^4} < 0,\\
&\psi_0(y) = dn(y; k)[1 - (1 + 2k^2 - \sqrt{1 - k^2 + 4k^4})sn^2(y; k)] > 0\\
&\\
&\mu_1 = 0\\
&\psi_1(y) = dn(y; k)sn(y; k)cn(y; k) = \tfrac{1}{2}\tfrac{d}{dy}cn^2(y; k)\\
&\\
&\mu_2 = k^2 - 2 + 2\sqrt{1 - k^2 + 4k^4} > 0\\
&\psi_2(y) = dn(y; k)[1 - (1 + 2k^2 + \sqrt{1 - k^2 + 4k^4})sn^2(y; k)].
\end{aligned}$$

Since the eigenvalues of L_+ and Λ are related by $\lambda_n = \beta\alpha^2\mu_n$, it follows that the first three eigenvalues of the operator L, equipped with periodic boundary condition on $[0, 2K(k)]$ are simple and $\lambda_0 < 0, \lambda_1 = 0, \lambda_2 > 0$. The corresponding eigenfunctions are $\psi_0(\alpha x), \psi_1(\alpha x) = const.\varphi'$ and $\psi_2(\alpha x)$.

Now we will verified that the condition (3) of Theorem 1 is satisfied in this case. From the expression of the operator L_+, we get $L_+[1] = c - 2\varphi$ and

$$1 = cL_+^{-1}[1] - 2L_+^{-1}\varphi. \tag{21}$$

Differentiating (7) with respect to c, we get

$$L_+\frac{d\varphi}{dc} + \varphi = \frac{dh}{dc} \tag{22}$$

and

$$\frac{d\varphi}{dc} + L_+^{-1}\varphi = \frac{dh}{dc}L_+^{-1}[1]. \tag{23}$$

From (21) and (23), we get

$$\left(c - 2\frac{dh}{dc}\right)\langle L_+^{-1}[1], 1\rangle = 2L - 2\langle\frac{d\varphi}{dc}, 1\rangle. \tag{24}$$

Since φ is mean zero, then $\langle\frac{d\varphi}{dc}, 1\rangle = \frac{d}{dc}\langle\varphi, 1\rangle = 0$. After integrating equation (7) and using that φ is mean zero, we get

$$2Th(c) = -\int_{-T}^{T}\varphi^2 dx. \tag{25}$$

From (18-20), after calculations, we obtain

$$\begin{aligned}
&\varphi_3 - \varphi_2 = 6\beta\kappa^2\tfrac{K^2(\kappa)}{T^2}\\
&\varphi_2 = -\tfrac{6\beta}{T^2}K(\kappa)[E(\kappa) - (1-\kappa^2)K(\kappa)\\
&\varphi_3 = \tfrac{6\beta}{T^2}[K^2(\kappa) - E(\kappa)K(\kappa)]\\
&c = \tfrac{4\beta}{T^2}[(2-\kappa^2)K^2(\kappa) - 3E(\kappa)K(\kappa)].
\end{aligned} \tag{26}$$

Note that the wave speed c needs to be a positive quantity by construction. On the other hand, this is not always satisfied - indeed, it is only true for some values of κ, please consult the Figure 1 below. It is clear that only values of $\kappa \in (0.98, 1)$ produce $c = c(\kappa) > 0$ as is required.

Using that

$$\int_0^{K(\kappa)} cn^2(x)dx = \frac{E(\kappa) - (1-\kappa^2)K(\kappa)}{\kappa^2}$$

and

$$\int_0^{K(\kappa)} cn^4(x)dx = \frac{1}{3\kappa^4}[(2-3\kappa^2)(1-\kappa^2)K(\kappa) + 2(2\kappa^2-1)E(\kappa)]$$

we obtain

$$\int_{-T}^{T}\varphi^2 dx = \frac{72\beta^2}{T^3}\left[-E^2(\kappa^2)K^2(\kappa) + \frac{2(2-\kappa^2)}{3}E(\kappa)K^3(\kappa) - \frac{1-\kappa^2}{3}K^4(\kappa)\right].$$

Using that $K'(\kappa) = \frac{E(\kappa)-(1-\kappa^2)K(\kappa)}{\kappa(1-\kappa^2)}$, $E'(\kappa) = E(\kappa) - K(\kappa)$ and differentiating the above expression with respect to c, we get

$$\frac{d}{dc}\int_{-T}^{T}\varphi^2 dx = \frac{436\beta^2}{T^3}\frac{[E^2(\kappa) - (1-\kappa^2)K(\kappa)][(2-\kappa^2)K(\kappa) - E(\kappa)K(\kappa)]}{\kappa(1-\kappa^2)}\frac{d\kappa}{dc}.$$

Now differentiating the fourth equation in (26) with respect to c, we get

$$1 = \frac{4\beta}{T^2}\frac{(2(2-\kappa^2)K(\kappa)E(\kappa) - 3E^2(\kappa) - (1-\kappa^2)K^2(\kappa)}{\kappa(1-\kappa^2)}\frac{d\kappa}{dc}.$$

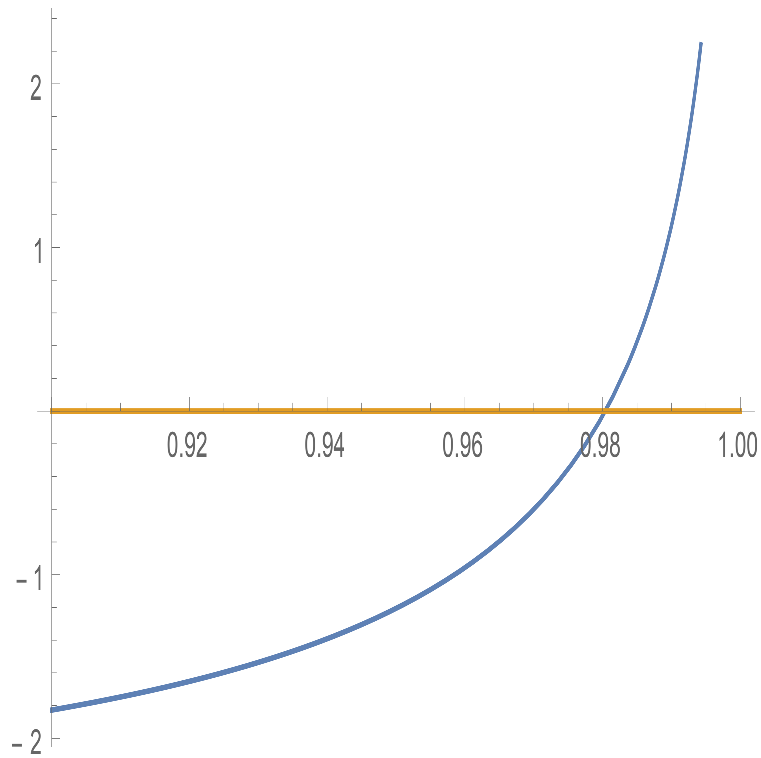

FIGURE 1. The graph of the function $(2-\kappa^2)K^2(\kappa)-3E(\kappa)K(\kappa)$ for $\kappa \in (0.9, 1)$

Combining all above expressions, we get

$$\frac{d\kappa}{dc} > 0, \quad \frac{d}{dc}\int_{-T}^{T} \varphi^2 dx > 0.$$

With this we verified that $\langle L_+^{-1}[1], 1\rangle \neq 0$.

3. Stability of traveling waves of the RSPE

We study the linear stability of the waves φ_ϵ, with respect of perturbations of the same period. *We assume that these solutions are even functions of x.*

3.1. **Spectral setup.** Consider the linearization around the solution $\varphi_\epsilon(x-ct)$ of (2), namely $u(t,x) = \varphi_\epsilon(x-ct) + v(t, x-ct)$. Note that since all solutions must have mean value zero, we take the perturbation $v \in L^2_0[-L, L]$ as well. After ignoring all terms of the form $O(v^2)$, we obtain

$$v_{tx} = (-\beta v_{xx} + cv - f'(\varphi_\epsilon)v - \epsilon\partial_x^{-2}v)_{xx}$$

By the mean value property of v, we can apply the operator ∂_x^{-1} to the previous identity to obtain

$$v_t = (-\beta v_{xx} + cv - f'(\varphi_\epsilon)v - \epsilon\partial_x^{-2}v)_x \tag{27}$$

Introduce the one-parameter family of operators

$$\mathcal{L}_\epsilon := -\beta\partial_{xx} + c - f'(\varphi_\epsilon) - \epsilon\partial_x^{-2}.$$

We denote the standard Hill operator corresponding to $\epsilon = 0$ by L, that is $L := \mathcal{L}_0$. Clearly, $\mathcal{L}_\epsilon$ is still self-adjoint, with domain H^2_0, which acts invariantly on the even and odd subspaces.

In order to rewrite the linear stability problem (27) in a more suitable form, introduce

$$w = |\partial_x|^{-1/2} v, w \in H^{5/2}_0$$

We then have, in terms of w (recall $\partial_x = -H|\partial_x|$)

$$|\partial_x|^{1/2} w_t = H|\partial_x|\mathcal{L}_\epsilon |\partial_x|^{1/2} w.$$

Applying $|\partial_x|^{-1/2}$ on both sides (and taking into account that H commutes with all $|\partial_x|^s$), we obtain

$$w_t = -H|\partial_x|^{1/2}\mathcal{L}_\epsilon|\partial_x|^{1/2} w =: -H\mathcal{K}_\epsilon w \tag{28}$$

This is the form of the eigenvalue problem considered in [12, 13]. In order to decide about the stability of (28), we need to establish some properties of the operators $\mathcal{L}_\epsilon, \mathcal{K}_\epsilon$. To begin with, observe that $\mathcal{K}_\epsilon$ is a self-asjoint operator with a domain $H^2_{per}[-L, L]$.

3.2. **Spectral analysis.**

Lemma 1. *Assume that the operator L has one simple negative eigenvalue, a simple eigenvalue at zero, corresponding to the eigenfunction φ_0' and the rest of the spectrum is strictly inside $(0, \infty)$. In addition, we require that $\langle L^{-1}[1], 1\rangle \neq 0$.*

Then, there exists $\epsilon_1 > 0$, so that the self-adjoint operator $\mathcal{K}_\epsilon$ is either a non-negative operator or else it has one simple negative eigenvalue. In addition, it has a simple eigenvalue at zero, corresponding to the eigenfunction $q_\epsilon = H|\partial_x|^{1/2}\varphi_\epsilon$, while the rest of the spectrum belongs to $(0, \infty)$.

Proof. We start our considerations with the operators $\mathcal{L}_\epsilon$. We have that

$$\mathcal{L}_\epsilon = \mathcal{L}_0 + f'(\varphi_0) - f'(\varphi_\epsilon) - \epsilon\partial_x^{-2} = L + M_\epsilon,$$

where M_ϵ is clearly a bounded operator, with $\|M_\epsilon\|_{L^2\to L^2} \leq C\epsilon$. Thus, by Courant principle for the eigenvalues[7], we can conclude that $|\lambda_j(\mathcal{L}_\epsilon) - \lambda_j(L)| \leq C\epsilon$.

Thus, for all small enough ϵ, we have one negative eigenvalue for $\mathcal{L}_\epsilon$ (which is close to $\lambda_0(L) < 0$). The eigenvalue $\lambda_2(\mathcal{L}_\epsilon)$ is in fact positive, being close to $\lambda_2(L) > 0$. The only remaining question is then about the sign of $\lambda(\mathcal{L}_\epsilon)$, which is close to $\lambda_1(L) = 0$. By direct verification (differentiating the equation (8) in x), we have that $\mathcal{L}_\epsilon(\varphi_\epsilon') = 0$, 0 is an eigenvalue and thus, we have that $\lambda_1(\mathcal{L}_\epsilon) = 0$. To summarize,

$$\lambda_0(\mathcal{L}_\epsilon) < 0 = \lambda_1(\mathcal{L}_\epsilon) < \lambda_2(\mathcal{L}_\epsilon).$$

We now need to make similar arguments regarding $\mathcal{K}_\epsilon$. Let ψ_ϵ be the negative eigenfunction for $\mathcal{L}_\epsilon$. According to the Courant variational eigenvalue principle, we have

$$\lambda_1(\mathcal{K}_\epsilon) \geq \inf_{\phi\perp|\partial_x|^{1/2}\psi_\epsilon} \frac{\langle \mathcal{K}_\epsilon\phi, \phi\rangle}{\|\phi\|^2}.$$

The restriction in the inf is equivalent to $\langle|\partial_x|^{1/2}\phi, \psi_\epsilon\rangle = 0$. But then,

$$\langle \mathcal{K}_\epsilon\phi, \phi\rangle = \langle \mathcal{L}_\epsilon|\partial_x|^{1/2}\phi, |\partial_x|^{1/2}\phi\rangle \geq 0,$$

[7]Here, $\lambda_j(S)$ refers to the j^{th} eigenvalue of the self-adjoint operator S, the smallest one being $\lambda_0(S)$

since $\mathcal{L}_\epsilon|_{\{\psi_\epsilon\}^\perp} \geq 0$. Thus, we conclude that $\lambda_1(\mathcal{K}_\epsilon) \geq 0$. This last conclusion implies that either $\mathcal{K}_\epsilon \geq 0$ or else it has at most one simple and negative eigenvalue.

Next, we study $Ker(\mathcal{K}_\epsilon)$. We will show that it is a one dimensional subspace, spanned by $q_\epsilon = H|\partial_x|^{1/2}\varphi_\epsilon$. To that end, let $\mathcal{K}_\epsilon\phi = |\partial_x|^{1/2}\mathcal{L}_\epsilon|\partial_x|^{1/2}\phi = 0$. As a consequence

$$\mathcal{L}_\epsilon|\partial_x|^{1/2}\phi = c, \tag{29}$$

where c could be zero or a non-zero constant. We show that it must be that $c = 0$ for all small enough ϵ. Indeed, assume for a moment that $c \neq 0$. Then, since $c \perp Ker(\mathcal{L}_\epsilon) = span\{\varphi'_\epsilon\}$, we can take inverses in (29) and

$$|\partial_x|^{1/2}\phi = c\mathcal{L}_\epsilon^{-1}[1].$$

But now, take a dot product of this last expression with the constant 1. We have

$$0 = \langle|\partial_x|^{1/2}1, \phi\rangle = \langle 1, |\partial_x|^{1/2}\phi\rangle = c\langle 1, \mathcal{L}_\epsilon^{-1}[1]\rangle.$$

Since $\lim_{\epsilon\to 0}\langle 1, \mathcal{L}_\epsilon^{-1}[1]\rangle = \langle 1, L^{-1}[1]\rangle \neq 0$, it follows that $c = 0$, a contradiction. Thus, $\mathcal{L}_\epsilon|\partial_x|^{1/2}\phi = 0$. However, recall that $\mathcal{L}_\epsilon$ has unique eigenfunction at zero, φ'_ϵ. Thus, $|\partial_x|^{1/2}\phi = const.\partial_x\varphi_\epsilon$ or

$$\phi = const.|\partial_x|^{-1/2}\partial_x\varphi_\epsilon = const.H|\partial_x|^{1/2}\varphi_\epsilon.$$

Finally, we need to show that the rest of the spectrum of $\mathcal{K}_\epsilon$ is in $(0,\infty)$. Indeed, we have already showed that $\lambda_1(\mathcal{K}_\epsilon) = 0$. If $\lambda_2(\mathcal{K}_\epsilon) > 0$, we are done. Otherwise, we would have $\lambda_2(\mathcal{K}_\epsilon) = 0$, which means that zero is an eigenvalue of multiplicity of at least two for $\mathcal{K}_\epsilon$. We have of course ruled this out in the argument above, so $\lambda_2(\mathcal{K}_\epsilon) > 0$ and Lemma 1 is proved in full.

□

For the stability of the waves, we use the results of [12, 13]. According to Lemma 1, we have that $\mathcal{K}_\epsilon$ is either a positive operator or else, it has one negative and simple eigenvalue. If $\mathcal{K}_\epsilon \geq 0$, we have that[8] $n(-H\mathcal{K}_\epsilon) \leq n(\mathcal{K}_\epsilon) = 0$ and hence the wave is stable. Otherwise, $n(\mathcal{K}_\epsilon) = 1$ and we have by [13] that

$$n(-H\mathcal{K}_\epsilon) = n(\mathcal{K}_\epsilon) - n(\langle\mathcal{K}_\epsilon^{-1}Hq_\epsilon, Hq_\epsilon\rangle) = 1 - n(\langle\mathcal{K}_\epsilon^{-1}Hq_\epsilon, Hq_\epsilon\rangle).$$

Thus, need to compute the quantity

$$\langle\mathcal{K}_\epsilon^{-1}Hq_\epsilon, Hq_\epsilon\rangle = \langle|\partial_x|^{-1/2}\mathcal{L}_\epsilon^{-1}|\partial_x|^{-1/2}[|\partial_x|^{1/2}\varphi_\epsilon], |\partial_x|^{1/2}\varphi_\epsilon\rangle = \langle\mathcal{L}_\epsilon^{-1}\varphi_\epsilon, \varphi_\epsilon\rangle.$$

Since both $\mathcal{L}_\epsilon$ and L act invariantly and are invertible on the subspace of even functions, we take the operator norms in the space of L^2_{even}. We have

$$\|\mathcal{L}_\epsilon^{-1} - L^{-1}\| = \|((I + L^{-1}M_\epsilon)^{-1} - I)L^{-1}\| \leq \|L^{-1}\|\|((I + L^{-1}M_\epsilon)^{-1} - I\|$$

It follows that $\|\mathcal{L}_\epsilon^{-1} - L^{-1}\| \leq C\epsilon$. Furthermore, $\|\varphi_\epsilon - \varphi\| \leq C\epsilon$ and hence

$$\lim_{\epsilon\to 0}\langle\mathcal{L}_\epsilon^{-1}\varphi_\epsilon, \varphi_\epsilon\rangle = \langle L^{-1}\varphi, \varphi\rangle$$

[8]Here $n(S)$ denotes the number of negative eigenvalues of the self-adjoint operator S

However, if we take a derivative with respect to c in (7), we see that $L[\varphi_c] = -\varphi$, whence $L^{-1}[\varphi] = -\frac{d\varphi}{dc}$. and hence

$$\langle L^{-1}\varphi, \varphi\rangle = -\langle \frac{d\varphi}{dc}, \varphi\rangle = -\frac{1}{2}\frac{d}{dc}\|\varphi\|^2.$$

Thus, if $\frac{1}{2}\frac{d}{dc}\|\varphi\|^2 > 0$, we find that $\lim_{\epsilon\to 0}\langle \mathcal{L}_\epsilon^{-1}\varphi_\epsilon, \varphi_\epsilon\rangle < 0$ and hence, $n(\langle \mathcal{L}_\epsilon^{-1}\varphi_\epsilon, \rangle) = 1$ for all small enough ϵ.

3.3. **The main stability result.** We summarize the existence and stability findings for φ_ϵ in the following result.

Theorem 2. *Assume that the nonlinearity f, the even solution φ of* (7) *and the operator $L_+ = -\beta\partial_x^2 + c - f'(\varphi)$ satisfy the assumptions of Theorem 1. Then, there exists $\epsilon_0 > 0$, so that travelling waves φ_ϵ exist for all $0 < \epsilon < \epsilon_0$.The functions φ_ϵ are even.*

Furthermore, assume that L_+ has a simple and single negative e-value and a simple eigenvalue at zero (with kernel spanned by φ'). Then, under the assumption

$$\frac{1}{2}\frac{d}{dc}\|\varphi\|^2 > 0$$

we can conclude that all waves φ_ϵ are linearly stable, when perturbed with perturbations with the same period.

By a way of example, we direct the reader to examine the properties of the periodic travelling wave solutions of KdV, constructed in Section 2.1. Clearly, all the requirements of Theorem 1 have been met Suppose φ is the periodic travelling wave for KdV provided in (19). Then, by Theorem 1, there exists $\epsilon_0 > 0$, so that for all $\epsilon : |\epsilon| < \epsilon_0$, there exist travelling wave solutions φ_ϵ of the regularized short pulse equation. By checking again the properties of φ, we conclude that these are spectrally stable, according to Theorem 2. We can thus formulate the following corollary.

Corollary 1. *Let φ be given* (19)*, with $\kappa \in (0.98, 1)$. Then, for all small enough ϵ, the corresponding travelling waves φ_ϵ of the regularized SPE are spectrally stable.*

4. Periodic travelling waves for the short-pulse equation

The generalized Ostrovsky equation that we consider is in the form (3), with $f(u) = -u^p$. For the purposes of this section, we consider the case $p = 2$ only, although some of the other cases are certainly physically relevant and mathematically tractable.

For the profile equation, we impose the ansatz $u(x,t) = \varphi_c(x + ct), c > 0$. We get

$$c\varphi'' = \varphi + (\varphi^2)_{xx} \tag{30}$$

or

$$[\varphi'(c - 2\varphi)]' = \varphi. \tag{31}$$

At this point, we need to perform a well-known change of variables, see for example [7], although this originated much earlier, see [2, 20, 23, 27, 29, 30]. More specifically, take

$$\xi = \eta - \frac{2\Psi(\eta)}{c} =: \Xi(\eta), \tag{32}$$

where $\varphi(\xi)=\Phi(\eta)=\Psi'(\eta)$. Then the equation (32), in the new variables takes the form

$$c^2\Phi_{\eta\eta}=\Phi(c-2\Phi). \tag{33}$$

Integrating once the equation (33), we get

$$\Phi_\eta^2=\frac{1}{c^2}\left[-\frac{4}{3}\Phi^3+c\Phi^2+A\right]=F(\Phi), \tag{34}$$

where A is a constant of integration. We have the following proposition.

Proposition 1. *There exists $a_0>0$, so that for every $a:|a|<a_0$, there exists an even smooth function Φ_a, solving (34), which is in the form*

$$\begin{aligned}\Phi_a(\eta) &= \frac{c}{2}+a\cos(k_a\eta)+O(a^2):-\frac{2\pi}{k_a}\le\eta\le\frac{2\pi}{k_a}\\ k_a &= \frac{1}{c}-\frac{10a^2}{3c^2}+O(a^4).\end{aligned}$$

Proof. In the phase plane (Φ,Φ') for $c>0$ equation (33) has equilibra at $(0,0)$ which is saddle point and at $\left(\frac{c}{2},0\right)$ which is a center. Let $\Phi_0<\frac{c}{2}<\Phi_1$ are roots of polynomial $F(\Phi)$. Then the period T of Φ is given by

$$T=\int_0^T dt=2\int_{\Phi_0}^{\Phi_1}\frac{1}{F(X)}dX. \tag{35}$$

It is easy to see that the period T is continuous function of Φ_0, indeed by changing of variable

$$X=\frac{\Phi_1-\Phi_0}{2}s+\frac{\Phi_1+\Phi_0}{2}$$

and for $\Phi_0\to\frac{c}{2}$, we get

$$T_\Phi\to T_0=2\pi\sqrt{c}. \tag{36}$$

Let $\Phi(x)=\phi(k_ax)=\phi(z)$, where $\phi(z)$ is periodic function with period 2π. To conform with our earlier setup, we will take the interval in the form $[-\pi,\pi]$. We have

$$k_0=\frac{2\pi}{T_0}=\frac{1}{\sqrt{c}}.$$

Then the function ϕ is given by the equation

$$c^2k_a^2\phi''=c\phi-2\phi^2. \tag{37}$$

Let

$$\begin{aligned}&\phi(z)=\tfrac{c}{2}+a\cos(z)+a^2\phi_2(z)+a^3\phi_3(z)+O(a^4),\\ &k_a^2=\tfrac{1}{c}+a^2k_1+O(a^4).\end{aligned} \tag{38}$$

We look for solutions parametrized by a small parameter a. Replacing in the equation(37), it is easy to see that coefficients of $O(a)$ is equal. For coefficients of $O(a^2)$, the compatibility condition is

$$c(\phi_2''+\phi_2)=-2\cos^2(z),$$

with solution $\phi_2(z) = \frac{1}{3c}(\cos(2z) - 3)$. For $O(a^3)$, we have

$$c(\phi_3'' + \phi_3) = c^2 k_1 \cos(z) - 4\cos(z)\phi_2(z),$$

which leads to the equality $k_1 = -\frac{10}{3c^2}$. Thus finally we get

$$\phi(z) = \frac{c}{2} + a\cos(z) + a^2 \frac{1}{3c}(\cos(2z) - 3) + a^3 \phi_3(z) + O(a^4), \tag{39}$$

and

$$k_a^2 = \frac{1}{c} - a^2 \frac{10}{3c^2} + O(a^4). \tag{40}$$

□

Now, to construct the solutions to (31), we need to use the function Φ described in Proposition 1. In addition, we need to make sure that we restrict ourselves to an appropriate interval, where the transformation (32) is invertible. To that end, take the derivative $\Xi'(\eta)$. We have

$$\frac{d\xi}{d\eta} = \Xi'(\eta) = 1 - \frac{2\Psi'(\eta)}{c} = 1 - \frac{2\Phi(\eta)}{c}.$$

From (38), we deduce that

$$\frac{d\xi}{d\eta} = \Xi'(\eta) = 1 - \frac{2}{c}(\frac{c}{2} + a\cos(k_a\eta) + O(a^2)) = -\frac{2a}{c}\cos(k_a\eta) + O(a^2).$$

Thus, we need to restrict η to an interval, where the $\cos(k_a\eta)$ does not vanish, for example

$$-\frac{B}{k_a} \le \eta \le \frac{B}{k_a}.$$

where $0 < B < \frac{\pi}{2}$. Now that we know that $\Xi(\eta) : [-B/k_a, B/k_a] \to \mathbf{R}^1$ has an inverse function, say $\eta(\xi)$, we need to determine the domain of the inverse function. Observing that

$$\begin{aligned} \Xi'(\eta) &= -\frac{2a}{c}\cos(k_a\eta) + O(a^2) < 0, \text{ when } \; a > 0 \\ \Xi'(\eta) &= -\frac{2a}{c}\cos(k_a\eta) + O(a^2) > 0, \text{ when } \; a < 0 \end{aligned}$$

Thus, we define the positive quantity ξ_a as follows

$$\begin{aligned} -\xi_a &= \Xi\left(\frac{B}{k_a}\right) \text{ when } \; a > 0 \\ \xi_a &= \Xi\left(\frac{B}{k_a}\right) \text{ when } \; a < 0 \end{aligned}$$

With this in mind, we just take

$$\varphi(\xi) := \phi(k_a\eta(\xi)), \varphi : [-\xi_a, \xi_a] \to \mathbf{R}^1, \tag{41}$$

where $\eta(\xi)$ is the inverse function to the one defined in (32).

By construction, φ solves (31). We now verify the periodicity in the intreval $[-\xi_a, \xi_a]$. We have

$$\varphi(\xi_a) = \phi(k_a\eta(\xi_a)) = \phi(-B), \varphi(-\xi_a) = \phi(k_a\eta(-\xi_a)) = \phi(B).$$

Thus, $\varphi(\xi_a) = \varphi(-\xi_a)$, since ϕ is an even function.

Furthermore,

$$\varphi'(\xi) = k_a\phi'(k_a\eta(\xi))\eta'(\xi) = k_a\frac{\phi'(k_a\eta(\xi))}{\frac{d\xi}{d\eta}} = k_a\frac{\phi'(k_a\eta(\xi))}{1-\frac{2}{c}\varphi(\xi)}.$$

Evaluating at $\xi = \pm\xi_a$, we have that

$$1-\frac{2}{c}\varphi(\pm\xi_a) = 1-\frac{2}{c}\phi(\mp B) = 1-\frac{2}{c}(\frac{c}{2}+a\cos(\mp B)+O(a^2)) = -\frac{2a}{c}\cos(B)+O(a^2) \neq 0$$

whence, the denominator is non-zero for small enough a. Also, for the numerator, we have

$$\phi'(k_a\eta(\xi_a)) = \phi'(-B) = -\phi'(B) = -\phi'(k_a\eta(-\xi_a)),$$

since ϕ' is an odd function. We conclude that

$$\varphi'(-\xi_a) = -\varphi'(\xi_a).$$

That is, the function has a corner crest at the endpoints of the interval $\pm\xi_a$. If one takes a periodization of such function, the corner crests will of course appear at all points in the form $(2k+1)\xi_a, k \in \mathcal{Z}$. This is the peakon type solution that we have discussed.

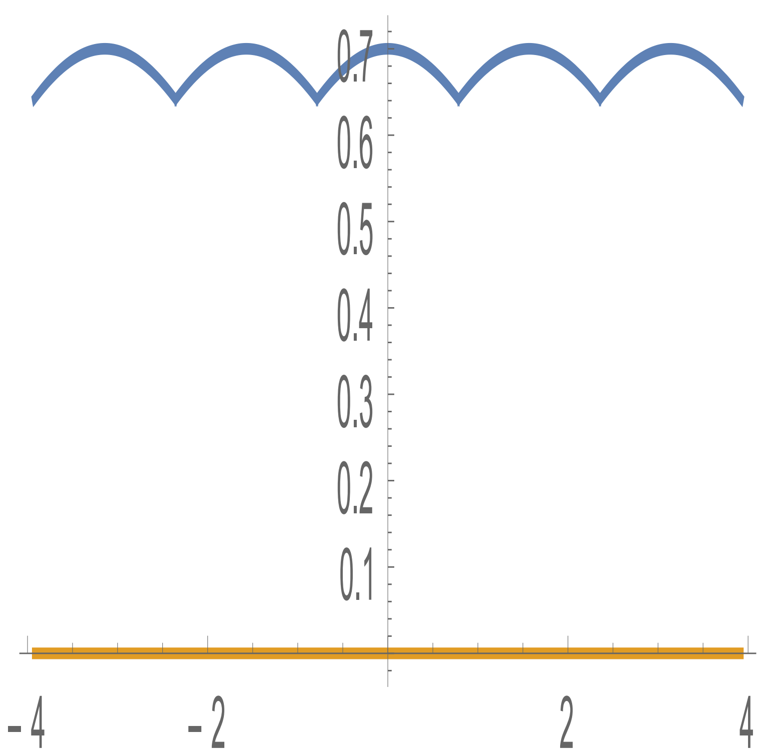

FIGURE 2. Peakons with $a > 0$

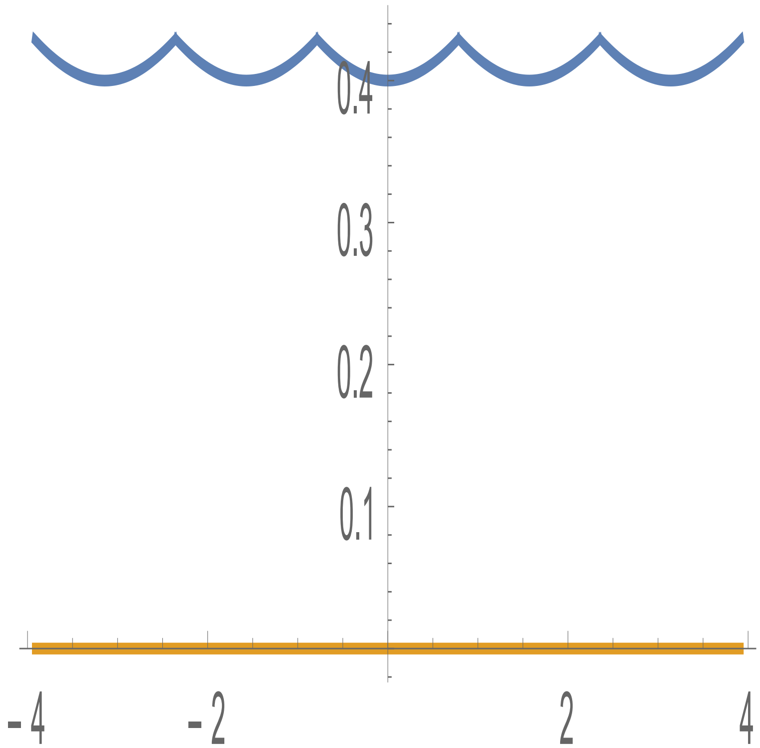

FIGURE 3. Peakons with $a < 0$

Theorem 3. *There exists $a_0 > 0$, so that for each $a : |a| < a_0$, there is a one parameter family of functions $\varphi_{a,b}, \varphi_{a,B} : [-\xi_a, \xi_a] \to \mathbf{R}^1$, $B \in (0, \pi/2)$, which classically solve* (31) *in the interval $(-\xi_a, \xi_a)$. These functions are $C^\infty(-\xi_a, \xi_a)$.*

Moreover, the $2\xi_a$ periodization of these functions remains a continuous function, but their derivatives develop jump discontinuities at the points $(2k+1)\xi_a, k \in \mathcal{Z}$.

5. SPECTRAL STABILITY OF THE TRAVELLING PEAKONS FOR THE SHORT PULSE EQUATION

In this section, we show the spectral stability for the peakons constructed in Section 4. The change of variables (32), which "translates" the peakon equation (31) into the more tractable Schrödinger equation (32) will play pivotal role in the problem for linear/spectral stability as well.

More concretely, suppose that φ_c is a solution[9] of (31) in say some interval $[-T, T]$. Introduce the ansatz $u(t,x) = \varphi_c(x+ct) + v(t, x+ct)$ in the equation (30). Ignoring quadratic and higher order terms, and adopting the new variable $x + ct \to \xi$, we arrive at

$$(v_t + ((c - 2\varphi)v)_\xi)_\xi = v \qquad -T < \xi < T, \tag{42}$$

[9]In the sense of Theorem 3

where v is a periodic function in $[-T,T]$. We consider the stability problem for (42). More precisely, take $v(t,\xi)=e^{\lambda t}w(\xi)$ so that (42) turns into the eigenvalue problem

$$(\lambda w+((c-2\varphi)w)_\xi)_\xi = w, \qquad -T<\xi<T. \tag{43}$$

Motivated by the form of the spectral problem (43), we look for w in the form $w=z_\xi$. Note that in doing so, we may and do assume that z is a $2T$ periodic function[10], so that $\int_{-T}^{T} z(\xi)d\xi=0$. Thus,

$$(\lambda z_\xi+((c-2\varphi)z_\xi)_\xi)_\xi = z_\xi, \qquad -T<\xi<T.$$

Integrating once in ξ allows us to conclude that

$$\lambda z_\xi+((c-2\varphi)z_\xi)_\xi = z+C, \qquad -T<\xi<T.$$

for some constant of integration C. But since $\int_{-T}^{T} z(\xi)d\xi=0$ and z is $2T$ periodic, we conclude that $C=0$, which leads to

$$\lambda z_\xi+((c-2\varphi)z_\xi)_\xi = z, \qquad -T<\xi<T. \tag{44}$$

By our construction of the peakons, the interval $[-T,T]$ is such that the transformation $\xi=\Xi(\eta)$ is a one-to-one mapping from $[-B/k_a,B/k_a]$ onto $[-T,T]$. Introduce $b:=\frac{B}{k_a}$. Take $\eta(\xi):[-T,T]\to[-b,b]$ to be the inverse of $\Xi(\eta)$. Thus, we may introduce a new function Z, via

$$z(\xi)=Z(\eta(\xi)).$$

We now compute the derivatives of z in terms of Z. We have

$$z_\xi(\xi)=\frac{Z_\eta}{\frac{d\xi}{d\eta}}=\frac{Z_\eta}{1-\frac{2}{c}\varphi(\xi)}=\frac{cZ_\eta}{c-2\varphi(\xi)}.$$

Thus, $(c-2\varphi)z_\xi=cZ_\eta$ and hence

$$[(c-2\varphi)z_\xi]_\xi = c\frac{d}{d\xi}(Z_\eta(\eta(\xi)))=c\frac{Z_{\eta\eta}}{\frac{d\xi}{d\eta}}=\frac{c^2 Z_{\eta\eta}}{c-2\varphi(\xi)}.$$

Plugging these expressions into (44), taking into account that $\varphi(\xi)=\Phi(\eta)$ and some algebraic manipulations leads us to the new spectral problem

$$-c^2 Z_{\eta\eta}+cZ-2\Phi Z=\lambda c Z_\eta, -b\le\eta\le b. \tag{45}$$

Let us show now that the corresponding z will have mean zero (as required per our construction), whenever $Z\in H^2$ is a solution to (45). Indeed, Z is smooth inside $(-b,b)$ and satisfy the equation (45). Thus, we have

$$\begin{aligned}\int_{-T}^{T} z(\xi)d\xi &= \int_{-T}^{T} Z(\eta(\xi))d\xi=\int_{-b}^{b} Z(\eta)\frac{d\xi}{d\eta}d\eta=\int_{-b}^{b} Z(\eta)\left(1-\frac{2}{c}\Phi(\eta)\right)d\eta= \\ &= c^{-1}\int_{-b}^{b}(c^2 Z_{\eta\eta}+\lambda c Z_\eta)d\eta.\end{aligned}$$

[10]This in fact fixes an unique function z with the property $w=z_\xi$

Now by Sobolev embedding, $Z \in H^2_{per}(-b,b)$ implies that (the $2b$ periodization of) both Z, Z_η are continuous at $\pm b$. As a consequence,

$$\begin{aligned}\int_{-b}^{b}(c^2 Z_{\eta\eta}+\lambda c Z_\eta)d\eta &= \lim_{\epsilon\to 0+} c^2(Z_\eta(b-\epsilon)-Z_\eta(-b+\epsilon))+\\ &+ \lim_{\epsilon\to 0+}\lambda c(Z(b-\epsilon)-Z(-b+\epsilon))=0.\end{aligned}$$

Since we are looking for solutions of (45) with $\Re\lambda>0$ and $c>0$, we can rewrite the spectral problem in the form

$$\mathcal{L}[Z]=\mu Z' \tag{46}$$

where $\mu=\lambda c$ and

$$\mathcal{L}:=-c^2\partial_{yy}+c-2\Phi, -b\le y\le b.$$

where $Z\in D(\mathcal{L})=H^2_{per}[-b,b]$. Clearly, the instability occurs for (45), exactly when it occurs for (46), because $\Re(\mu)=c\Re(\lambda)$ and hence $\Re(\mu)$ and $\Re(\lambda)$ have the same sign. In other words, we study (46) for stability/instability.

Let $P_0 : L^2[-b,b]\to L^2_0[-b,b]$ be the projection onto L^2_0

$$P_0 f(y)=f(y)-\frac{1}{2b}\int_{-b}^{b} f(z)dz.$$

Introduce the operator $\mathcal{L}_0 := P_0\mathcal{L}P_0$, which is self-adjoint, when equipped with the domain $D(\mathcal{L}_0)=H^2_0[-b,b]$. Alternatively, one may define it via the quadratic form $q(u,v)=\langle \mathcal{L}u,v\rangle$, where $u,v\in H^1_0[-b,b]$. We will need the following lemma regarding the spectrum of $\mathcal{L}$.

Lemma 2. *The spectrum of $\mathcal{L}$ consists of eigenvalues only, each with finite multiplicity. There exists $a_0>0$, so that for all $-a_0<a<0$, the operator $\mathcal{L}_a$ is a positive operator. For all $a: 0<a<a_0$, $\mathcal{L}_a$ has one negative eigenvalue - $\lambda_0(\mathcal{L})$, $\frac{|\lambda_0(\mathcal{L}_a)|}{a}\sim 1$ and the rest of the spectrum is positive. In particular, $\mathcal{L}_a$ is invertible for all $a: |a|<a_0$. Finally, there exists $\delta=\delta_b$, so that $\mathcal{L}_0\ge \delta Id$.*

Proof. Clearly, the spectrum of $\mathcal{L}$ consists entirely of real eigenvalues, each with finite multiplicity. This is because $\mathcal{L}$ is bounded from below and also $\mathcal{L}$ can be represented as a sum of $A=-\partial_{yy}$ plus a relative compact perturbation. We have by (39)

$$\begin{aligned}\mathcal{L} &= -c^2\partial_{yy}+c-2\Phi=-c^2\partial_{yy}+c-2\left(\frac{c}{2}+a\cos(k_a y)+O(a^2)\right)=\\ &= -c^2\partial_{yy}-2a\cos(k_a y)+O(a^2).\end{aligned}$$

Recall that $y\in(-b,b)$ in which $\cos(k_a y)>0$. In fact, $\cos(k_a y)\ge\sigma_b>0$, whence

$$\mathcal{L}\ge -c^2\partial_{yy}-2a\sigma_b\ge -2a\sigma_b=2|a|\sigma_b.$$

Thus, we immediately conclude that for all $a<0, |a|<<1$, we have that $\mathcal{L}$ is a positive operator.

For $a>0$, we observe that

$$\mathcal{L}=-c^2\partial_{yy}-2a\cos(k_a y)+O(a^2)\ge -2a\cos(k_a y)+O(a^2)\ge -2a,$$

whence $\lambda_0(\mathcal{L}) \geq -2a$. On the other hand, using the expression for Φ from Proposition 1

$$\langle \mathcal{L}(1), 1\rangle = \langle c - 2\Phi, 1\rangle = -2a\langle 1, \cos(k_a\cdot)\rangle + O(a^2) \leq -ab\sigma_b, \tag{47}$$

since $\cos(k_a y) \geq \sigma_b > 0, y \in (-b, b)$.

Finally, restrict to the mean value zero subspace $L^2_0[-b, b]$, which is co-dimension one subspace. We have for $f \in H^1_0[-b, b]$

$$\langle \mathcal{L}f, f\rangle = c^2\|f'\|^2 - 2a\int_{-b}^{b} f^2(y)\cos(k_a y)dy + O(a^2)$$

On the other hand, for $f \in H^1_0[-b, b]$, there is the Poincare inequality, $\|f'\| \geq \frac{\pi}{b}\|f\|$. Thus, we conclude that for all $a : |a| << 1$, and for all for $f \in H^1_0[-b, b]$

$$\langle \mathcal{L}f, f\rangle \geq \left(\frac{c^2\pi^2}{b^2} - 2a + O(a^2)\right)\|f\|^2 \geq \frac{c^2\pi^2}{2b^2}\|f\|^2.$$

Note that this last inequality incidentally establishes the last claim in Lemma 2.

By the min-max characterization of the eigenvalues, the last inequality implies that $\lambda_1(\mathcal{L})$ is positive, bounded away from zero (in terms of a). Combining the last observations implies that for $0 < a << 1$, $\mathcal{L}$ has a small (and simple) negative eigenvalue and the rest of the spectrum is contained in (k_b, ∞), for some $k > 0$.

In addition, we can deduce that in fact the negative eigenvalue $|\lambda_0(\mathcal{L})| \sim a$. More precisely, there exist $0 < c_0 \leq 2$, so that for all $0 < a << 1$,

$$c_0 \leq \frac{|\lambda_0(\mathcal{L})|}{a} \leq 2. \tag{48}$$

Indeed, the right-hand side inequality follows from our earlier estimate $\lambda_0(\mathcal{L}) \geq -2a$, while the left-hand side inequality eventually follows from (47).

To that end, let $\psi_0, \{\psi_j\}_j$ is an enumeration of the eigenfunctions of $\mathcal{L}$, corresponding to eigenvalues $\lambda_0(\mathcal{L}) < 0$, $0 < c_b \leq \lambda_1(\mathcal{L}) \leq \lambda_2(\mathcal{L}) \ldots$. We have

$$-2ab\sigma_b \geq \langle \mathcal{L}(1), 1\rangle = \lambda_0(\mathcal{L})\langle 1, \psi_0\rangle^2 + \sum_{j=1}^{\infty} \lambda_j(\mathcal{L})\langle 1, \psi_j\rangle^2 \geq \lambda_0(\mathcal{L})\langle 1, \psi_0\rangle^2.$$

Noting that $|\langle 1, \psi_0\rangle| \leq \|1\|\|\psi_0\| = C$, we conclude that $-2ab\sigma_b \geq -C|\lambda_0(\mathcal{L})|$. But this last inequality means $|\lambda_0(\mathcal{L})| \geq c_0 a$, whence (48) is established in full.

$\square$

Having the results of Lemma 2, we return to the consideration of the eigenvalue problem (46). Introduce a new variable $W : W = \mathcal{L}[Z]$ or $Z = \mathcal{L}^{-1}[W]$, since $\mathcal{L}$ is invertible. This allows us to rewrite (46) as

$$W = \mu\partial_x[\mathcal{L}^{-1}W]. \tag{49}$$

Since the right-hand side of the last equation is an exact derivative, W has mean value zero or $W \in L^2_0[-b, b]$. In fact, keeping track of the derivatives[11] allows us to conclude

[11]By definition $\mathcal{L}^{-1}[W] \in H^2[-b, b]$, whence $\partial_x\mathcal{L}^{-1}[W] \in H^1[-b, b]$

$W \in H^1_0[-b,b]$. Thus, we introduce yet another new variable $Q \in H^{3/2}_0[-b,b]$, with $W = |\partial_x|^{1/2} Q$. Taking into account $\partial_x = -H|\partial_x|$, we can rewrite (49) as follows

$$-H|\partial_x|^{1/2}\mathcal{L}^{-1}|\partial_x|^{1/2}Q = \mu^{-1}Q. \tag{50}$$

Observe that in (50) $\mathcal{L}^{-1}$ acts upon $|\partial_x|^{1/2}Q \in L^2_0[-b,b]$ and then, the operator $|\partial_x|^{1/2}$ in front of $\mathcal{L}^{-1}$ can be factorized $|\partial_x|^{1/2} = |\partial_x|^{1/2}P_0$. Thus, we may rewrite (50)

$$-H|\partial_x|^{1/2}\mathcal{L}_0^{-1}|\partial_x|^{1/2}Q = \mu^{-1}Q. \tag{51}$$

Note that in this particular form, the eigenvalue problem fits the framework of [12, 13]. Indeed, with $J := -H$ and $L := |\partial_x|^{1/2}\mathcal{L}_0^{-1}|\partial_x|^{1/2}$, we have a pair of anti self-adjoint and self-adjoint operators, so that the eigenvalue problem of interest is in the form $JLQ = \mu^{-1}Q$. Moreover, we have that $L > 0$, since for every $f \in H^2_0[-b,b]$

$$\langle Lf, f\rangle = \langle \mathcal{L}_0^{-1}|\partial_x|^{1/2}f, |\partial_x|^{1/2}f\rangle > 0,$$

since $\mathcal{L}_0 > \delta Id$ by virtue of Lemma 2. Thus, the operator $L > 0$. Based on that, we can now conclude that the eigenvalue problem $JLQ = hQ$ cannot have solutions with $\Re h > 0$, which implies that (50) and subsequently (46) does not have unstable solutions. This follows by a simple application of the general results in [12, 13] (and maybe as a consequence of some older papers). Let us however provide a simple alternative proof.

Assume that $JLQ = hQ$ has unstable solutions, that is

$$JL(Q_1 + iQ_2) = (h_1 + ih_2)(Q_1 + iQ_2).$$

with $h_1 > 0$. Taking real and imaginary parts yields the system

$$\left| \begin{array}{l} JLQ_1 = h_1 Q_1 - h_2 Q_2 \\ JLQ_2 = h_2 Q_1 + h_1 Q_2 \end{array} \right.$$

Taking dot products with LQ_1, LQ_2 respectively and taking into account $\langle Jf, f\rangle = 0$ for all real-valued functions[12] , we obtain

$$\left| \begin{array}{l} h_1\langle Q_1, LQ_1\rangle - h_2\langle Q_2, LQ_1\rangle = 0 \\ h_2\langle Q_1, LQ_2\rangle + h_1\langle Q_2, LQ_2\rangle = 0. \end{array} \right.$$

By the self-adjointness of L, $\langle Q_2, LQ_1\rangle = \langle Q_1, LQ_2\rangle$. Thus, adding the equations results in

$$h_1(\langle Q_1, LQ_1\rangle + \langle Q_2, LQ_2\rangle) = 0.$$

Recall though that $L > 0$, whence $(\langle Q_1, LQ_1\rangle + \langle Q_2, LQ_2\rangle > 0$, whence $h_1 = 0$, a contradiction.

We have thus proved the main result.

Theorem 4. *There exists $a_0 > 0$, so that for all $a : |a| < a_0$, the waves φ_a constructed in Section 4 are spectrally stable.*

[12]This property is of course valid for all skew symmetric operators J, which map real-valued into real-valued functions

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