- **Problem 1**
  Solve the initial value problem explicitly and sketch the graph of the solution

$$\begin{cases} y' = \frac{-x}{1+y} \\ y(0) = 1 \end{cases}$$

What is the domain of the solution?

$\boxed{5 \text{ points}}$

This is a separable equation.

$$\frac{dy}{dx} = \frac{-x}{1+y}$$

$$\int -x\,dx = \int (1+y)\,dy$$

$$-\frac{x^2}{2} = y + \frac{y^2}{2} + c$$

$$\boxed{y^2 + 2y + x^2 + 2c = 0} \quad \boxed{10 \text{ points}}$$

$y(0) = 1$ means $1^2 + 2.1 + 0^2 + 2c = 0$

$$\boxed{2c = -3}$$

The implicit solution is given by

$$\boxed{y^2 + 2y + x^2 - 3 = 0} \quad \boxed{5 \text{ points}}$$

Solve as a quadratic equation for $y$:

$$y_{1,2} = -1 \pm \sqrt{1 - x^2 + 3} = -1 \pm \sqrt{4 - x^2}$$

Since $y(0) = 1$, we have to choose „+" sign.

The explicit solution is given by $\boxed{y(t) = -1 + \sqrt{4 - x^2}}$

Domain: $4 - x^2 \geq 0$, $\boxed{-2 \leq x \leq 2}$ $\boxed{5 \text{ points}}$ $\boxed{10 \text{ points}}$

Graph:

$\boxed{5 \text{ points}}$

- **Problem 2**

Solve the equation implicitly

$$(3x^2 + y^2) + (2xy - 6y^2)y' = 0$$

10 points

$$\left.\begin{aligned} M(x,y) &= 3x^2+y^2 \\ N(x,y) &= 2xy-6y^2 \end{aligned}\right\} \quad \left.\begin{aligned} M_y &= 2y \\ N_x &= 2y \end{aligned}\right\} \Rightarrow \boxed{\text{Exact equation}}$$

We will look for $\varphi(x,y)$ such that

$$\varphi_x = M(x,y) = 3x^2+y^2$$

$$\varphi_y = N(x,y) = 2xy-6y^2$$

Thus $\varphi(x,y) = \int (3x^2+y^2)\,dx =$

$$= \boxed{x^3 + xy^2 + f(y)}$$ 20 points

Differentiate $\varphi(x,y)$ in $y$ to get

$$\varphi_y(x,y) = 2xy + f'(y) = 2xy - 6y^2$$

$$f'(y) = -6y^2 \Rightarrow f(y) = -2y^3$$

thus $\boxed{\varphi(x,y) = x^3 + xy^2 - 2y^3}$ 10 points

The implicit solution is given by

$$\boxed{x^3 + xy^2 - 2y^3 = c}$$

- **Problem 3** Find the solution of the given initial value problem and describe the behavior of the solution for large $t$.

$$y'' - 2y' + 5y = 0, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 2$$

Describe the behavior of the solution as $t \to \infty$.

Write the characteristic equation:

$$\boxed{r^2 - 2r + 5 = 0} \quad \boxed{5 \text{ points}}$$

$$r_{1,2} = 1 \pm \sqrt{1-5}$$

$$\boxed{r_{1,2} = 1 \pm 2i} \quad \boxed{5 \text{ points}}$$

The solution is given by

$$\boxed{y(t) = c_1 \cdot e^t \cdot \cos 2t + c_2 \cdot e^t \sin 2t} \quad \boxed{10 \text{ points}}$$

We will determine the value of $c_1$, $c_2$.

$$y\left(\tfrac{\pi}{2}\right) = c_1 \cdot e^{\frac{\pi}{2}} \cdot \cos \pi + c_2 \cdot e^{\pi/2} \cdot \sin \pi = -c_1 \cdot e^{\pi/2} = 0$$

$$\Rightarrow \boxed{c_1 = 0} \quad \boxed{5 \text{ points}}$$

$$y(t) = c_2 \cdot e^t \cdot \sin 2t$$

$$y'(t) = c_2 e^t \sin 2t + 2 c_2 e^t \cos 2t$$

$$y'\left(\tfrac{\pi}{2}\right) = -2 c_2 e^{\pi/2} = 2 \quad \Rightarrow \boxed{c_2 = -e^{-\pi/2}} \quad \boxed{5 \text{ points}}$$

$$\boxed{y(t) = -e^{-\pi/2} \cdot e^t \cdot \sin 2t} \quad \boxed{5 \text{ points}}$$

As $t \to \infty$, the solution exibits $\boxed{\text{growing amplitude oscillations.}}$ $\boxed{5 \text{ points}}$

- **Problem 4** Solve the initial-value problem

$$y'' + y' - 2y = 2t, \quad y(0) = 0, \quad y'(0) = 1$$

The characteristic equation for the homogeneous ODE is given by

$$\boxed{r^2 + r - 2 = 0}$$ 5 points

$$r_{1,2} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2}$$

$\boxed{\text{real roots: } r_1 = 1, \ r_2 = -2}$ 5 points

The solution of the homogeneous equation is given by

$\boxed{y_h(t) = c_1 . e^{t} + c_2 . e^{-2t}}$ 5 points

Next, find $Y(t)$ - a particular solution of the nonhomogeneous equation. Seek $\boxed{Y(t) = At + B}$ 5 points

Then $Y'(t) = A$, $Y''(t) = 0$, the equation gives

$$0 + A - 2(At + B) = 2t$$
$$-2At + A - 2B = 2t \implies \begin{array}{l} -2A = 2, \ \boxed{A = -1} \\ A - 2B = 0 \ \ \boxed{B = -\frac{1}{2}} \end{array}$$

$\boxed{Y(t) = -t - \frac{1}{2}}$ 10 points

The general solution is given by

$\boxed{y(t) = c_1 . e^{t} + c_2 . e^{-2t} - t - \frac{1}{2}}$ 5 points

Compute the constants from the initial conditions

$\boxed{y(t) = e^{t} - \frac{1}{2} e^{-2t} - t - \frac{1}{2}}$ 5 points

- **Problem 5**
  Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 L of a dye solution with a concentration of 1 g/L. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in a rate of 2 L/min, the well-stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches 1% of its original value.

$Q(t)$ – the amount of dye at time $t$

$Q(0) = 200 \times 1 = 200g$

$Q'(t) =$ rate in – rate out

$$Q'(t) = 0 \times 2 - \frac{Q(t)}{200} \times 2$$

20 points

$$\boxed{\text{IVP} \quad \begin{cases} Q' = -\frac{1}{100} Q \\ Q(0) = 200 \end{cases}}$$

→ linear or separable equation

Solve for $Q$ to get: $\boxed{Q(t) = 200 \cdot e^{-1/100 \cdot t}}$

10 points

1% of 200 = 2 g of dye solution.

$200 \cdot e^{-T/100} = 2$ → Solve for $T$.

$$e^{-\frac{T}{100}} = \frac{1}{100}$$

$$\ln\left(e^{-\frac{T}{100}}\right) = -\frac{T}{100} = \ln\frac{1}{100} = -\ln 100$$

$$\Rightarrow \boxed{T = 100 \ln 100 \approx}$$

10 points

- **Extra Credit Problem (20 points)** Solve the initial-value problem with a parameter $b$.

$$y'' - y' + 0.25y = 0, \quad y(0) = 2, \quad y'(0) = b$$

Find the critical value of $b$ that separates solutions that always remain positive from those that eventually become negative.

Characteristic equation is: $r^2 - r + \frac{1}{4} = 0$

double roots: $\boxed{r_{1,2} = \frac{1}{2}}$

$$\boxed{y(t) = c_1 \cdot e^{t/2} + c_2 \cdot t \cdot e^{t/2}}$$

$$y(0) = c_1 = 2$$

$$y'(0) = 1 + c_2 = b \Rightarrow \boxed{c_2 = 1 - b}$$

$$\boxed{y(t) = 2e^{t/2} + (1-b)\, t \cdot e^{t/2}}$$

If $1 - b < 0$, the solution $y(t)$ eventually will be $< 0$.

$\Rightarrow \boxed{b = 1}$ is the critical value.

Full Credit or No credit only for the bonus problem.