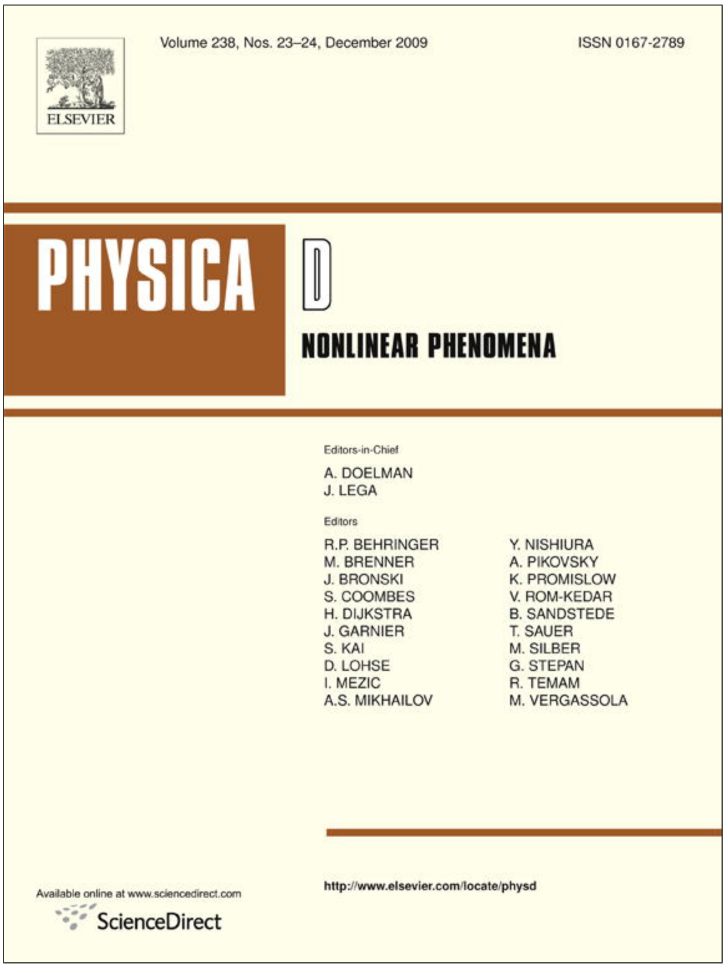
Volume 238, Nos. 23–24, December 2009
ISSN 0167-2789
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Physica D 238 (2009) 2298–2307

Contents lists available at ScienceDirect

Physica D

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# On precise center stable manifold theorems for certain reaction–diffusion and Klein–Gordon equations

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ARTICLE INFO

*Article history:*
Received 27 April 2009
Received in revised form
9 September 2009
Accepted 14 September 2009
Available online 24 September 2009
Communicated by B. Sandstede

*Keywords:*
Stable manifold
Center manifold
Klein–Gordon equation

ABSTRACT

We consider positive, radial and exponentially decaying steady state solutions of the general reaction–diffusion and Klein–Gordon type equations and present an explicit construction of infinite-dimensional invariant manifolds in the vicinity of these solutions. The result is a precise stable manifold theorem for the reaction–diffusion equation and a precise center-stable manifold theorem for the Klein–Gordon equation, which include the co-dimension of the manifolds and the decay rates for even perturbations.

Published by Elsevier B.V.

## 1. Introduction

In this paper, we will be interested in the stability properties of special solutions of a fairly general class of reaction–diffusion and Klein–Gordon type systems. More precisely, these will feature linearly unstable steady states, which are nevertheless physically interesting objects. The conditional stability of such nonlinearly unstable solutions will be our main topic here. In a recent paper by Karageorgis and Strauss [1], the authors have shown that the spectrally unstable ground states are indeed non-linearly unstable as well. We will informally refer to such statements as "linear instability $\Rightarrow$ nonlinear instability" theorems. We point out that such statements are by no means automatic. In fact, their proofs almost always require the construction of quite clever functionals of the solution, which blow up if the initial data is selected close to the unstable special solution. The blow up happens either in finite time or at infinity, along the evolution. One of the main examples in the paper [1] is the equation $u_{tt} - \Delta u = |u|^5, x \in R^3$, which is known to admit a family of steady state solutions $\varphi_\lambda(x) = \frac{(3\lambda^2)^{1/4}}{\sqrt{\lambda^2+|x|^2}}$, $\lambda > 0$. Using the general theorem, it is proved that these solutions are linearly and nonlinearly unstable.

On the other hand, in another recent paper of interest, [2] by Krieger and Schlag consider the same equation, with initial data close to the steady state solution $\varphi_1$, which is unstable with a simple unstable mode $-\sigma^2$. The authors construct a manifold $\Sigma$ such that if the perturbation to $\varphi_1$, $(\psi_0, \psi_1) \in \Sigma$, then the solution exists globally and remains near $\varphi_1$ for all $t > 0$. The tangent plane to $\Sigma$ is given by

$$\sigma \int_{R^n} \xi(x)\psi_o(x)\mathrm{d}x + \int_{R^n} \xi(x)\psi_1(x)\mathrm{d}x = 0,$$

where $\xi$ is the eigenvector corresponding to the unstable eigenvalue of the linearized operator. The condition for nonlinear instability in [1] is that $\sigma \int_{R^n} \xi(x)\psi_o(x)dx + \int_{R^n} \xi(x)\psi_1(x)dx > 0$. Thus, the co-dimension one manifold $\Sigma$, constructed in [2], separates the dispersive solutions from those that blow up in finite time. In essence, Krieger and Schlag prove a conditional stability result, which provides a condition for the solutions to stay close to the steady state for all $t > 0$. There is in fact a conjecture made in [2], stating that for initial data "below" the stable manifold $\Sigma$, the solutions scatter to zero. We refer the reader to the interesting paper [3] for numerical simulations and further discussion in support of this phenomenon.

Inspired by these recent works (see also [29]), we were able to prove similar conditional stability result for the radial steady states of the Klein–Gordon equation in space dimensions 2, 3 and 4 with respect to even perturbations. In the language of invariant manifolds, we construct an explicit co-dimension one center-stable manifold for the Klein–Gordon equation. In the classical paper of Bates and Jones [4], it was proved that there exists nontrivial unstable manifold for the radial perturbations of the radial steady state solutions of the Klein–Gordon equation $u_{tt} - \Delta u + u - |u|^{p-1}u = 0$, $(t, x) \in \mathbf{R}^1_+ \times \mathbf{R}^d$ if $d \geq 3, p < \frac{d}{d-2}$. These finite-dimensional stable and unstable manifolds are of

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0167-2789/$ – see front matter. Published by Elsevier B.V.
doi:10.1016/j.physd.2009.09.010

equal dimensions and the center manifold is infinite–dimensional. Moreover, in [5] the same authors show that the steady states are stable with respect to radial perturbations restricted to the center manifold which, in essence, proves the uniqueness of the center manifold. There exist an arbitrary small neighborhoods of the solution for which any initial data that does not lie on the center manifold must leave that neighborhood in forward or backward time. We want to emphasize that these positive, radial and exponentially decaying steady state solutions do exist for $p < \frac{d+2}{d-2}$, when $d \geq 3$ and for any $p$, when $d = 1, 2$, [6] (see [22,23,26] as well). In our paper, we were able to prove the existence of co-dimension one center stable manifold for the case of dimensions 2, 3 and 4 and any power in the regime $\frac{d+2}{d-2} > p > 1 + \frac{4}{d}$, for which the steady state solutions exist, and in addition the linearized operator $\mathcal{H}$ satisfies the gap condition (H4), described below. In addition, there is an orthogonality condition which gives the conditional stability result exactly as in the paper by Krieger and Schlag [2]. Our techniques are closer in spirit to [2], but the results complement those of Bates and Jones, [4]. It is of interest to consider general perturbations for the Klein–Gordon equation instead of even or radial ones. This case is more complicated due to the multiplicity of the eigenvalue at zero. Our initial computations show that, in this case, the co-dimension of the center-stable manifold will be strictly bigger than one.

Since our goal is to prove conditional stability type results, in essence we do not distinguish too much between the center and center stable manifolds, both of which contribute to solutions that do not grow exponentially in time. Similar ideas were exploited recently in [7] to prove conditional stability of nonclassical (Lax and undercompressive) viscous shock waves of general quasilinear parabolic systems of conservation laws.

In order to make our ideas precise and to state the results, let us outline the framework. The problems that will be considered will be either reaction–diffusion models in the form

$$u_t - \Delta u + u - \mathcal{N}(u) = 0 \quad (t, x) \in \mathbf{R}^1_+ \times \mathbf{R}^d \tag{1}$$

or Klein–Gordon type equations of the type

$$u_{tt} - \Delta u + u - \mathcal{N}(u) = 0 \quad (t, x) \in \mathbf{R}^1_+ \times \mathbf{R}^d. \tag{2}$$

We consider first the simpler reaction–diffusion case (1) as a motivation and in order to set up the notations and the ideas in the proof, and then proceed to formulate and prove the theorems for the main equation of interest (2). In the next section we formulate our assumptions and their consequences for the spectrum of the corresponding linearized operators.

### 1.1. Standing assumptions

We need the following assumptions on the nonlinearity $\mathcal{N}$, which will be assumed to hold henceforth.

(H1) The equation $-\Delta\phi + \phi - \mathcal{N}(\phi) = 0$ has a solution $\phi$, which is positive, smooth, radial and $\lim_{x\to\pm\infty} \partial_x^\alpha \phi(x) e^{a|x|} = 0$ for some $a > 0$ and for any $\alpha \in \mathcal{Z}^d$.

(H2) The linearized operator $\mathcal{H} = -\Delta + 1 - \mathcal{N}'(\phi)$ has a negative eigenvalue.

(H3) $\mathcal{N}(0) = \mathcal{N}'(0) = 0$ and for each compact set $K \subset \mathbf{R}^d$, $\mathcal{N} \in C^{2+\alpha}(K)$ for some $0 < \alpha \leq 1$. In particular, there exists $C_K$, so that for each $x, x + h \in K$,

$$|\mathcal{N}(x + h) - \mathcal{N}(x) - h\mathcal{N}'(x)| \leq C_K |h|^{1+\alpha},$$
$$|\mathcal{N}'(x + h) - \mathcal{N}'(x) - h\mathcal{N}''(x)| \leq C_K |h|^{1+\alpha},$$

Note that (H3) implies that for every $N > 0$, there exists $C_N$, so that $|\mathcal{N}'(s)| \leq C_N s^\alpha$ for all $0 < s < N$.

Let us take the opportunity to review the assumptions. First, (H1) is a natural assumption for the existence of a ground state for (1). Simple sufficient conditions are presented in Theorem 3.1, [8]. The assumption (H3) helps to ensure that $\mathcal{N}(u)$ is indeed a nonlinear term. Slightly less restrictive variants of these are all present in Karageorgis–Strauss, [1] assumptions (A1)–(A4). The assumption (H2) reflects our goal to study unstable configurations. Below, we discuss two natural situations, in which (H2) is satisfied.

The first is when the function $\mathcal{N}$ is convex, which is the set up[1] in [1]. This is well-known to cause the appearance of unstable point spectrum, as the following elementary computation shows. Indeed, note that since $\mathcal{H}\phi = \mathcal{N}(\phi) - \mathcal{N}'(\phi)\phi = \mathcal{N}(\phi) - \mathcal{N}(0) - \mathcal{N}'(\phi)\phi$

$$\langle \mathcal{H}\phi, \phi\rangle = \int (\mathcal{N}(\phi) - \mathcal{N}(0) - \mathcal{N}'(\phi)\phi)\phi \mathrm{d}x$$
$$= \int \left(\int_0^1 (\mathcal{N}'(z\phi) - \mathcal{N}'(\phi))\mathrm{d}z\right)\phi^2 \mathrm{d}x.$$

Now, the convexity of $\mathcal{N}$ implies that $\lambda \to \mathcal{N}'(\lambda)$ is increasing and since $\phi > 0$, it follows that $\int_0^1 (\mathcal{N}'(z\phi) - \mathcal{N}'(\phi))dz < 0$. Thus

$$\inf \sigma(\mathcal{H}) = \inf_{\|\chi\|=1} \langle \mathcal{H}\chi, \chi\rangle \leq \frac{1}{\|\phi\|^2} \langle \mathcal{H}\phi, \phi\rangle < 0. \tag{3}$$

Hence $\sigma_{p.p.}(\mathcal{H}) \supseteq \sigma(\mathcal{H}) \cap (-\infty, 0) \neq \emptyset$.

Another situation[2], where an instability occurs is when $d \geq 2$ and the steady state $\phi$ is radial and decaying. Then

$$\mathcal{H}(\phi) = (-\Delta + 1 - \mathcal{N}'(\phi))\phi'(\rho) = -(d - 1)\rho^{-2}\phi'(\rho),$$

implying $\langle \mathcal{H}[\phi'], \phi'\rangle < 0$, whence

$$\inf \sigma(\mathcal{H}) = \inf_{\|\chi\|=1} \langle \mathcal{H}\chi, \chi\rangle \leq \frac{1}{\|\phi'\|^2} \langle \mathcal{H}[\phi'], \phi'\rangle < 0.$$

Regarding the a.c. spectrum of $\mathcal{H}$, since we have $\lim_{|x|\to\infty} \mathcal{N}'(\phi(x)) = 0$, it follows by Weyl's theorem [9,27] that $\sigma_{a.c.}(\mathcal{H}) = \sigma_{a.c.}(-\Delta + 1) = [1, \infty)$. Finally, by $|\mathcal{N}'(s)| \leq C_N s^\alpha$ for all $s < N$, we have that the potential satisfies $|\mathcal{N}'(\phi)| \leq C_\phi \phi^\alpha$, whence by the exponential decay of $\phi$ (as required in (H1)), we conclude that $\mathcal{H}$ has finitely many eigenvalues, an absolute continuous spectrum (with no embedded eigenvalues) and no other spectrum. Note also that by differentiating the identity $-\Delta\phi + \phi - \mathcal{N}(\phi) = 0$, we see that $\mathcal{H}[\partial_j\phi] = 0, j = 1, \ldots, d$. That is, $\{\partial_j\phi\}$ are linearly independent (and in fact mutually orthogonal) eigenvectors, corresponding to the eigenvalue zero. In fact, we can arrange so that all eigenfunctions be mutually orthogonal. Also, without loss of generality, we assume that all (except for $\{\partial_j\phi\}$) are of $L^2$ norm one. Next, we collect our conclusions.

**Proposition 1.** *Let the nonlinearity $\mathcal{N}(u)$ satisfy hypotheses (H1)–(H3). Then the linearized operator $\mathcal{H}$ has the following spectrum*

$$\sigma(\mathcal{H}) = \{-\sigma_1^2, \ldots, -\sigma_M^2\} \cup \{0\} \cup \{\mu_1^2, \ldots, \mu_N^2\} \cup [1, \infty) :$$
$$\sigma_{a.c.}(\mathcal{H}) = [1, \infty),$$

*where $M \geq 1$, $N \geq 0$ and $\mu_j^2 < 1, j = 1, \ldots, N$. Moreover, the eigenvalue zero is of multiplicity at least d. The eigenfunctions $\{\psi_j\}_{j=1}^M$ and $\{\chi_j\}_{j=1}^N$ (corresponding to the eigenvalues $\{-\sigma_j^2\}_{j=1}^M$ and $\{\mu_j^2\}_{j=1}^N$ respectively) and $\{\partial_j\phi\}$, corresponding to the eigenvalues at zero are smooth, decaying at infinity and mutually orthogonal.*

In other words, for all examples satisfying (H1)–(H3), the ground state $\phi$ is linearly unstable and hence nonlinearly unstable as well, following Karageorgis–Strauss, [1]. Moreover, there may be at most finitely many, precisely $M$ in our notations, unstable directions.

[1] Of course, it has to be recognized that Karageorgis and Strauss are working in a more general setting, but in spirit, the convexity of $\mathcal{N}$ guarantees the linear instability.

[2] We thank an anonymous referee for generously pointing out this.

### 1.2. Some more spectral results

While the elementary considerations in the previous section shed some light on the structure of the spectrum of $\mathcal{H}$, we shall need more detailed information, especially in the case of the Klein–Gordon equation, (2).

Regarding the eigenvalue at zero, it is well-known that at least some of its eigenvectors arise out of symmetries for the problem. Thus, 0 is at least a multiplicity $d$ eigenvalue, due to translation invariance. We would like to emphasize that this problem was studied in great detail in connection with linearizations of Nonlinear Schrödinger equations around pulse solutions. Indeed, our operator $\mathcal{H}$ is exactly the classical operator $L_+$, which appears in the NLS context (see [24,25,30]). The following result is due to Weinstein, [10] in the cases $d = 1$ and $d = 3$ and in the general case to Kwong, [6], see also Lemma 2.1 in [11] for a new and elegant proof.

**Proposition 2** *([10,6,11]). Let $d \geq 1$ and $\mathcal{N}(u) = u^p$, where*

$$1 < p < \begin{cases} \infty & d = 1, 2 \\ (d+2)/(d-2) & d \geq 3. \end{cases}$$

*Then $\mathrm{Ker}(\mathcal{H}) = \{\partial_j \phi, j = 1, \ldots, d\}$. In general, if $\mathcal{N}$ satisfies the assumptions[3] of Theorem 3.1, [8], and if $\phi$ is decaying, then $\mathrm{Ker}(\mathcal{H}) \supseteq \{\partial_j \phi, j = 1, \ldots, d\}$, and moreover $\dim(\mathrm{Ker}(\mathcal{H})) = d$ or $\dim(\mathrm{Ker}(\mathcal{H})) = d + 1$.*

The last statement above is a consequence of an argument in Proposition 2.8, [10], which rules out all eigenvectors at zero (other than $\partial_j \phi, j = 1, \ldots, d$) with the possible exception of a purely radial eigenvector.

In addition, we will need another spectral property of our operator $\mathcal{H}$, namely the so-called *gap condition*, which is our hypothesis

(H4) $\mathcal{H}$ does not have eigenvalues in $(0, 1]$ and the point 1 is not a resonance.

Hypothesis (H4) was also a spectral assumptions of Schlag, [12] (and later Beceanu, [13]) in the proof of existence of center-stable manifolds for the pulses for the cubic focusing nonlinear Schrödinger equation in three dimensions. Unfortunately, such a statement is not rigorously available, to the best of our knowledge, for any $d \geq 2$ and $p > 1$. The case $d = 1$ is actually well-understood (see for example Theorem 3.1, [11]) for all $p > 1$. The conclusion there is that for $p \geq 3$, the operator $\mathcal{H}$ satisfies the gap condition.

Returning to the case $d = 3$, there is a *numerical justification of* (H4), *at least for the case $d = 3$ and $p$ close to 3.*

**Proposition 3** *(Demanet–Schlag, [14]). Let $d = 3$, $\beta_* = 0.914\ldots$ and $1 + 2\beta_* < p \leq 3$. Then $\mathcal{H} = -\Delta + 1 - p\phi^{p-1}$ does not have eigenvalues in $(0, 1]$ and 1 is not a resonance point.*

The authors, [14] claim that such a result is true for at least some values of $p$ bigger than 3. On the other hand, eigenvalues start to appear in $(0, 1)$ for $p < 1 + 2\beta_*$ as shown numerically in [14]. Note that some further numerical simulations for the case $d = 2, 3$ are available in [11], but they run only over possible eigenvectors in the subspace of the first few harmonics. Finally, in the case $d = 2$, P. Kevrekidis, [15] has recently informed us, that a series of numerical simulations showed that the operator $\mathcal{H}$ (or $L_+$) does not have eigenvalues in $(0, 1)$ at least for $p = 4$, $p = 5$.

### 1.3. Main results

Our main result is, in essence, a fairly explicit construction of an center–stable manifold $\Sigma$ in each of the two cases mentioned above. This is a conditional stability result which states that whenever the initial data $u_0$ satisfies $u_0 - \phi \in \Sigma$, then the solution will approach (in an exponential way or slower) a translate of the equilibrium solution.

As pointed out in the beginning, the center stable manifold for the Klein–Gordon equation is our main goal here, but to illustrate the method of the proof on a simpler case, we will consider first the reaction–diffusion equation as a model problem. In that case, due to the spectral gap, we have exponential decay estimates for $e^{-t\mathcal{H}} P_{a.c.}$. Consequently, we may construct stable manifold under very general conditions of the nonlinearity $\mathcal{N}$. We believe that the results that we obtain, while not new per se, nevertheless shed new light on the behavior close to the unstable steady state, even in this simple setting. In particular, one may extend formula (5) below for the asymptotic phase into an asymptotic expansion in powers of $\varepsilon$.

### 1.4. Reaction–diffusion case

**Theorem 1.** *For the Eq. (1), assume hypotheses (H1)–(H3) and 1 is neither a resonance nor an eigenvalue of $\mathcal{H}$. Fix $q_0 > \max(2, d)$.*

*Then, there exists $0 < \varepsilon = \varepsilon \ll 1$, $C > 0$ and $h = (h_1, \ldots, h_M) : B(L^2 \cap \dot{W}^{1,q_0}(\mathbf{R}^d), \varepsilon) \to \mathbf{R}^M$, so that whenever the initial data is in the form*

$$u(0) = \phi + f + \sum_{j=1}^{M} h_j(f)\psi_j : f \in span[\psi_1, \ldots, \psi_M]^\perp,$$
$$\|f\|_{L^2 \cap \dot{W}^{1,q_0}} < \varepsilon,$$

*then the solution $u$ can be written as*

$$u(t,x) = \phi(x + \vec{y}(t)) + \sum_{j=1}^{M} a_j(t)\psi_j + \sum_{j=1}^{d} b_j(t)\partial_j\phi$$
$$+ \sum_{j=1}^{N} c_j(t)\chi_j + z(t), \tag{4}$$

*where $\vec{y}(t) \in C^1([0,\infty)) : y(0) = 0, |y'(t)| \leq Ce^{-t}\varepsilon$, $y^\infty = \lim_{t\to\infty} \vec{y}(t)$.*

*Fix $0 < \sigma < \min(\sigma_1^2, \ldots, \sigma_M^2, \mu_1^2, \ldots, \mu_N^2; 1)$. Then, one has the decay estimates,*

$$\sup_{f : \|f\|_{L^2 \cap \dot{W}^{1,q_0}} \leq \varepsilon} |h(f)| \leq C\varepsilon^{1+\alpha},$$

$$\|z(t)\|_{L^2 \cap \dot{W}^{1,q_0}} + \sum_{j=1}^{d} |b_j(t)| + \sum_{j=1}^{M} |c_j(t)| \leq C\varepsilon e^{-t\sigma},$$

$$|\vec{y}(t) - y^\infty| \leq C\varepsilon e^{-t}, \quad \sum_{j=1}^{M} |a_j(t)| \leq C\varepsilon^{1+\alpha} e^{-t\sigma}.$$

*In addition,*

$$y_j^\infty = \frac{1}{\|\partial_j\phi\|^2}\langle f, \partial_j\phi\rangle + O(\varepsilon^{1+\alpha}) \tag{5}$$

*is an approximate formula for the asymptotic phase.*

We would like to first point out that the noneigenvalue/nonresonance condition of 1 here is likely to be removable, but we keep it for sake of simplicity of the presentation. Also, the existence of infinite–dimensional stable manifold and finite–dimensional unstable and center manifolds for the reaction diffusion systems is not a new fact. In fact, it was proved in [16] using center manifold reduction methods (see also [28] for recent and more sophisticated results). We believe however, that Theorem 1 gives more information than the corresponding results in [16,4].

In addition, the proof of Theorem 1 serves as an illustration of the setup and the techniques that we will use to treat the more subtle Klein–Gordon example.

[3] so that smooth, radial and decaying solutions $\phi$ exists

### 1.5. Klein–Gordon case

For the Klein–Gordon example (2), naturally we have more restrictive results. This is due mainly to the power (dimension-dependent) time decay of the semigroup generator $\cos(t\sqrt{H})P_{a.c.}(H)$ and this is especially problematic as the dimension gets smaller. In order to present the main ideas and the assumptions in a more straightforward manner, we will consider only the case of power nonlinearity in the form $\mathcal{N}(u) = |u|^{p-1}u$. One could potentially, at the expense of some technicalities, formulate and prove similar results for general nonlinearities, satisfying conditions similar to the ones in Theorem 1.

Another simplifying assumption, which we choose to make, is that *we consider only even initial data* for (2). This may not seem like much of a restriction, but it actually simplifies our problem a great deal. In particular, if one considers $\mathcal{H}$ on the subspace of even functions, it does not have 0 in its point spectrum and, therefore, it prevents a dangerous build-up of mass at the zero modes of the system. Our initial computations show that the stable manifold for the problem with general (i.e. not necessarily even) initial data is of co-dimension higher than one and it should in fact be $d^2 + 1$. The technical difficulties associated with such a theorem would be an order of magnitude beyond the ones that we present here. We hope to be able to report on this problem in a future publication.

Our main result concerning (2) is the explicit construction of a co-dimension one center stable manifold for the steady state solutions in space dimension two, three and four. For dimension one as well as dimensions higher than four, there are technical complications and we are not including these cases here. We prove the following theorem.

**Theorem 2.** *For (2), assume $\mathcal{N}(u) = |u|^{p-1}u$, where*

$$1 + 4/d \le p < \begin{cases} \infty & d = 2 \\ (d+2)/(d-2) & d = 3, 4. \end{cases}$$

*Then, there is $\sigma = \sigma(p)$, so that $-\sigma^2 \in \sigma_{p.p.}(\mathcal{H})$ with $\mathcal{H}\psi = -\sigma^2\psi$.*

*If in addition, we further restrict p, so that the gap condition (H4) holds. Let $s_d : s_3 = s_4 = 2, s_2 = 1$. Then there exists $0 < \varepsilon = \varepsilon(k, d) \ll 1$, $C = C(d) > 0$ and a function*

$$h : B_{H^{s_d}}(\varepsilon) \times B_{H^{s_d-1}}(\varepsilon) \cap \{(f, g) : \langle \sigma f + g, \psi\rangle = 0\} \to \mathbf{R}^1$$

*so that whenever the initial data is even and in the form*

$$\left| \begin{array}{l} u(0) = \phi + f_1 + h(f_1, f_2)\psi \\ u_t(0) = f_2 \end{array} \right. ;$$

$$\langle \sigma f_1 + f_2, \psi\rangle = 0; \ \|(f_1, f_2)\|_{H^{s_d}\times H^{s_d-1}} < \varepsilon,$$

*then the solution u can be written as*

$$u(t, x) = \phi(x) + a(t)\psi + \mathbf{z}(t), \tag{6}$$

*where $\mathbf{z} = P_{a.c.}(\mathcal{H})\mathbf{z}$ and*

- *For $d = 3, 4$, there exists C so that*

$$\begin{aligned} &\|z\|_{L^\infty_t[0,\infty)H^2_x \cap L^2_t[0,\infty)W_x^{3/2-1/d,2d/(d-2)}} \le C\varepsilon \\ &\|a\|_{L^1_t[0,\infty)\cap L^\infty_t[0,\infty)} \le C\varepsilon. \end{aligned} \tag{7}$$

- *For $d = 2$, fix $q_0 : 2 < q_0 < 8/3$, and $r_0 : 1/q_0 + 1/r_0 = 1/2$. Then, there exists C so that*

$$\begin{aligned} &\|z\|_{L^\infty_t[0,\infty)H^1_x \cap L^{q_0}_t[0,\infty)W_x^{1-2/q_0,r_0}} \le C\varepsilon \\ &\|a\|_{L^2_t[0,\infty)\cap L^\infty_t[0,\infty)} \le C\varepsilon. \end{aligned} \tag{8}$$

We take the opportunity to informally discuss the results of Theorem 2.

- We limit our considerations to the case $d \le 4$ for reasons of physical interest as well as certain technical problems that arise when the power $p < 2$. In other words, a similar theorem should hold true (with the same statement) for the KG equation in any $d \ge 5$, when $p \ge 2$ and modulo certain minor technical adaptations of the argument (like Proposition 4 below), when $p < 2$.
- Our smoothness assumptions on the data are not necessarily the sharpest possible, but we wanted to use integer values of $s_d$ to avoid some minor, but cumbersome technical issues.
- The case $d = 1$, on the other hand, is an important one from a physical point of view. There are, however, more serious technical complications here, that are essentially due to the lack of sufficient time decay of the KG semigroup. In order to deal with that, one needs to consider a further refinement of the function spaces considered in the proof of Theorem 2, which allow to close the argument in spaces with very little time decay. This case is currently under investigation.

A few words about the organization of the paper. In Section 2, we give the proof of Theorem 1. In Section 3, we present some technical tools to deal with the KG problem. Namely, we first derive the equations for the different components of the ansatz for the KG equation of Theorem 2. We also present the Strichartz estimates for the free KG equation (following [17]) and the technique of the wave operators, which allows us to state the Strichartz estimates (in Sobolev spaces) for the perturbed KG equation. In Section 4, we finally present the proof of Theorem 2.

## 2. Proof of Theorem 1

We will be looking for a solution $u$ in the form (4). More precisely, we will write differential equations for the unknown functions $a_j, b_j, c_j, j = 1, \dots, d$ and $z(t)$, which we will solve using fixed points for certain maps. We will show that these maps do indeed have fixed points, in view of the linear estimates that they satisfy. These will be, in turn, a consequence of the spectral assumptions and the decay of the bound state.

Note first that the space $L^2 \cap \dot{W}^{1,q_0} \subset L^\infty(\mathbf{R}^d)$ by Sobolev embedding, since $q_0 > d$. This implies that we will have *a priori* control of the $L^\infty_x$ norms of all pieces of the solution and hence, some of the constants will depend implicitly on that.

### 2.1. Derivation of the linearized equation

To set the ideas, write

$$\begin{aligned} &\mathcal{N}\Big(\phi(x+\vec{y}(t)) + \sum_{j=1}^M a_j(t)\psi_j + \sum_{j=1}^d b_j(t)\partial_j\phi + \sum_{j=1}^N c_j(t)\chi_j + z(t)\Big) \\ &\quad = \mathcal{N}(\phi(x+\vec{y}(t))) + \mathcal{N}'(\phi(x+\vec{y}(t)))\left(\sum_{j=1}^M a_j(t)\psi_j + \sum_{j=1}^d b_j(t)\partial_j\phi \right. \\ &\quad\quad \left. + \sum_{j=1}^N c_j(t)\chi_j + z(t)\right) + [\mathcal{N}(\phi+G) - \mathcal{N}(\phi) - \mathcal{N}'(\phi)G], \end{aligned}$$

where

$$G(t, x) = \sum_{j=1}^M a_j(t)\psi_j + \sum_{j=1}^d b_j(t)\partial_j\phi + \sum_{j=1}^N c_j(t)\chi_j + z(t).$$

Clearly, by the Hölder assumption (H3), we have the following estimates[4] for the error term

$$|\mathcal{N}(\phi+G) - \mathcal{N}(\phi) - \mathcal{N}'(\phi)G| \le C|G|^{1+\alpha},$$
$$|\nabla[N(\phi+G) - \mathcal{N}(\phi) - \mathcal{N}'(\phi)G]| \le C(|G|^{1+\alpha} + |\nabla G||G|^\alpha).$$

From here, an application of the Hölder's inequality and Sobolev embedding shows

[4] Note that the constant $C$ that appears is a consequence of (H3) and hence, it will depend on the $L^\infty$ bounds on $\phi$, $\nabla\phi$, but also on the *a priori* bound for $\|G\|_{L^\infty}$ that we tacitly assume.

$$\|\mathcal{N}(\phi+G)-\mathcal{N}(\phi)-\mathcal{N}'(\phi)G\|_{L^2\cap\dot W^{1,q_0}} \le C\|G\|^{1+\alpha}_{L^2\cap\dot W^{1,q_0}}.$$

Furthermore, again by the Hölder assumption (H3)

$$\mathcal{N}'(\phi(x+\vec{y}(t))) = \mathcal{N}'(\phi(x)) + O(|y(t)|).$$

To summarize, we have shown the representation formula

$$\begin{aligned}&\mathcal{N}\left(\phi(x+\vec{y}(t)) + \sum_{j=1}^M a_j(t)\psi_j + \sum_{j=1}^d b_j(t)\partial_j\phi + \sum_{j=1}^N c_j(t)\chi_j + z(t)\right)\\ &\quad = \mathcal{N}(\phi(x+\vec{y}(t))) + \mathcal{N}'(\phi(x))\left(\sum_{j=1}^M a_j(t)\psi_j + \sum_{j=1}^d b_j(t)\partial_j\phi\right.\\ &\quad\left. + \sum_{j=1}^N c_j(t)\chi_j + z(t)\right) + \Xi(t,x),\end{aligned}$$

where $\Xi$ is so that

$$\begin{aligned}\|\Xi(t,\cdot)\|_{L^2\cap\dot W^{1,q_0}} &\le C(|a(t)|^{1+\alpha} + |b(t)|^{1+\alpha} + |c(t)|^{1+\alpha}\\ &\quad + \|z(t)\|^{1+\alpha}_{L^2\cap\dot W^{1,q_0}} + C|y(t)|(|a(t)| + |b(t)| + |c(t)|\\ &\quad + \|z(t)\|_{L^2\cap\dot W^{1,q_0}})).\end{aligned}$$

In addition,

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(x+\vec{y}(t)) = \sum_{j=1}^d y_j'(t)\partial_j\phi(x) + O(|y(t)||y'|).$$

Taking into account the relations,

$$\begin{cases}-\Delta\phi + \phi - \mathcal{N}(\phi) = 0,\\ -\Delta\psi + \psi - \mathcal{N}'(\phi)\psi = \mathcal{H}\psi = -\sigma^2\psi\\ -\Delta\partial_j\phi + \partial_j\phi - \mathcal{N}'(\phi)\partial_j\phi = \mathcal{H}[\partial_j\phi] = 0,\end{cases} \tag{9}$$

we find that the evolution of $u$ is governed by the equation

$$\begin{cases} z_t + \mathcal{H}z + \sum_{j=1}^M \psi_j(a_j'(t) - \sigma_j^2 a_j(t)) + \sum_{j=1}^d (y_j'(t) + b_j'(t))\partial_j\phi + \sum_{j=1}^N (c_j'(t)\\ \quad + \mu_j^2 c_j(t))\chi_j + F(x, \vec{y}(t), z(t), a(t), b_1(t)\ldots, b_d(t)) = 0,\end{cases} \tag{10}$$

where $F = \Xi + O(|y(t)||y'|)$ and thus, satisfies similar estimate as $\Xi$.

This is a nonlinear equation for the scalar unknowns $a_j(t), j = 1,\ldots,M$; $b_j(t), j = 1,\ldots,d$; $c_j(t), j = 1,\ldots,N$ and the function $z(t)$.

Introduce the variable $m(t) = (y^\infty, \{a_j(t)\}_{j=1}^M, \{b_j(t)\}_{j=1}^d, \{c_j(t)\}_{j=1}^N, z(t))$ which lives in the space $X = \mathbf{R}^d \times C([0,\infty), \mathbf{R}^{M+d+N}) \times C([0,\infty), L^2_x \cap \dot W^{1,q_0})$. At this point, we make the assignment, $y_j(t) := (1-\mathrm{e}^{-t})y_j^\infty$, for the center of the traveling wave. To summarize, $F$ satisfies

$$\begin{aligned}\|F(t, m(\cdot))\|_{L^2\cap\dot W^{1,q_0}} &\le C(|\vec{a}(t)|^{1+\alpha} + |\vec{b}(t)|^{1+\alpha} + |\vec{c}(t)|^{1+\alpha}\\ &\quad + \|z(t)\|^{1+\alpha}_{L^2\cap\dot W^{1,q_0}} + |y^\infty|^2\mathrm{e}^{-t}) + C|y^\infty|(|a(t)|\\ &\quad + |b(t)| + |c(t)| + \|z(t)\|_{L^2\cap\dot W^{1,q_0}}).\end{aligned} \tag{11}$$

One has in fact more general Hölder's continuity estimate in the spirit of (11).

### 2.2. Setting the contraction map

In order to show the existence of these quantities for all times, we set up the solution mapping $\Lambda$ and show that it is a contraction[5]. Recall that $\|\psi_j\|_{L^2} = \|\chi_j\|_{L^2} = 1$. Taking a projection onto $\psi_j$ yields

$$a_j'(t) - \sigma_j^2 a_j(t) + \langle F(t), \psi_j\rangle = 0.$$

This gives the representation formula

$$\mathrm{e}^{-t\sigma_j^2}a_j(t) - a_j(0) + \int_0^t \mathrm{e}^{-s\sigma_j^2}\langle F(s), \psi_j\rangle\mathrm{d}s = 0.$$

Thus, if one expects $\lim_{t\to\infty} a_j(t) = 0$, we obtain

$$a_j(0) = \int_0^\infty \mathrm{e}^{-s\sigma^2}\langle F(s), \psi_j\rangle\mathrm{d}s, \tag{12}$$

whence going back to the formula for $a_j(t)$, we have

$$a_j(t) = \int_t^\infty \mathrm{e}^{-(s-t)\sigma^2}\langle F(s), \psi_j\rangle\mathrm{d}s. \tag{13}$$

Next, take a projection onto $\partial_j\phi$, whence

$$y_j'(t) + b_j'(t) + \langle F(t), \partial_j\phi/\|\partial_j\phi\|^2\rangle = 0.$$

Integrating in time and taking $t\to\infty$ (recall $y_j(0) = 0$, $\lim_{t\to\infty} b_j(t) = 0$) yields the formula

$$y_j^\infty = b_j(0) - \int_0^\infty \langle F(s), \partial_j\phi/\|\partial_j\phi\|^2\rangle\mathrm{d}s. \tag{14}$$

Since we need functions $y_j(t)$, so that $y_j(0) = 0$, $\lim_{t\to\infty} y_j(t) = y_j^\infty$, it is actually a good idea to assign $y_j(t) := (1-\mathrm{e}^{-t})y_j^\infty$. Thus, we obtain the following formula for $b_j(t)$

$$b_j(t) = \mathrm{e}^{-t}y_j^\infty + \int_t^\infty \langle F(s), \partial_j\phi/\|\partial_j\phi\|^2\rangle\mathrm{d}s, \tag{15}$$

where here $y_j^\infty$ is the dynamically defined variable in (14).

Regarding the scalar variables $c_j(t)$, they clearly have to satisfy $c_j(0) = \langle f, \chi_j\rangle$. Then projection onto $\chi_j$ yields

$$c_j(t) = \mathrm{e}^{-t\mu_j^2}\langle f, \chi_j\rangle - \int_0^t \mathrm{e}^{(s-t)\mu_j^2}\langle F(m(s)), \chi_j\rangle\mathrm{d}s. \tag{16}$$

Following this heuristic arguments, we build our map $\Lambda$ as follows. First, by (13), set

$$\tilde{a}_j(t) = \int_t^\infty \mathrm{e}^{-(s-t)\sigma_j^2}\langle F(m(s)), \psi_j\rangle\mathrm{d}s, \tag{17}$$

where $F(m(s))$ is the nonlinear expression from (10). Following (14) and (15), set

$$\tilde{y}_j^\infty = \frac{1}{\|\partial_j\phi\|^2}\langle f, \partial_j\phi\rangle - \int_0^\infty \langle F(m(s)), \partial_j\phi/\|\partial_j\phi\|^2\rangle\mathrm{d}s. \tag{18}$$

and

$$\tilde{b}_j(t) = \mathrm{e}^{-t}\tilde{y}_j^\infty + \int_t^\infty \langle F(m(s)), \partial_j\phi/\|\partial_j\phi\|^2\rangle\mathrm{d}s,$$

Finally

$$\tilde{z}(t) = \mathrm{e}^{-t\mathcal{H}}P_{a.c.}(\mathcal{H})f + \int_0^t \mathrm{e}^{-(t-s)\mathcal{H}}P_{a.c.}(\mathcal{H})F(m(s))\mathrm{d}s. \tag{19}$$

With this, we have defined $\tilde{m} = \Lambda[m]$. We now need to show that such a map is a contraction on an appropriate space. For the variable $m(t) = (y^\infty, \{a_j(t)\}_{j=1}^M, \{b_j(t)\}_{j=1}^d, \{c_j(t)\}_{j=1}^N, z(t))$, which lives in $\mathbf{R}^d\times C([0,\infty), \mathbf{R}^{M+d+N})\times C([0,\infty), L^2_x\cap\dot W^{1,q_0})$, introduce the quantities

$$M_0(m) = |\vec{y}^\infty|$$

$$M_1(m) = \sup_{0\le t<\infty} \mathrm{e}^{t\sigma}\max_{1\le j\le M}|a_j(t)|$$

$$M_2(m) = \sup_{0\le t<\infty} \mathrm{e}^{t\sigma}\max_{1\le j\le d}|b_j(t)|$$

$$M_3(m) = \sup_{0\le t<\infty} \mathrm{e}^{t\sigma}\max_{1\le j\le N}|c_j(t)|$$

$$M_4(m) = \sup_{0\le t<\infty} \mathrm{e}^{t\sigma}\|z(t)\|_{L^2_x\cap\dot W^{1,q_0}},$$

[5] We will follow the convention $\tilde{m} = \Lambda(m)$, that is we denote the new variables as a tilda version of the old variables.

where $0 < \sigma < \min(\sigma_1^2, \dots, \sigma_M^2, \mu_1^2, \dots, \mu_N^2, 1)$ is some fixed number in $(0, 1)$.

Clearly, the expression $\max(M_0(m), \dots, M_4(m))$ is a norm on a subset of $\mathbf{R}^d \times C([0, \infty), \mathbf{R}^{M+d+N}) \times C([0, \infty), L_x^2 \cap \dot{W}^{1,q_0})$, which we denote by $X$ and $\|m\|_X := \max(M_0(m), \dots, M_4(m))$. Note that this norm guarantees rate of time decay of at least $e^{-t\sigma}$ for all functions $\vec{a}(t), \vec{b}(t), \vec{c}(t), \|z(t)\|_{L_x^2 \cap \dot{W}^{1,q_0}}$.

We now show that there exists $0 < \delta, \varepsilon \ll 1$, such that whenever $\|f\|_{L^2} \le \delta\varepsilon$, then $\Lambda : B(X, \varepsilon) \to B(X, \varepsilon)$ is a contraction. Note first, that from (11)

$$\|F(m(t))\| \le C\|m\|_X^{1+\alpha} e^{-t\sigma}. \tag{20}$$

Then, for some constants $C = C_d$, we estimate in (18)

$$M_0(\tilde{m}) = \max_{1\le j\le d} |\tilde{y}_j^\infty| \le \max_{1\le j\le d} \left( \frac{1}{\|\partial_j\phi\|}\|f\| + \frac{1}{\|\partial_j\phi\|}\int_0^\infty \|F(m(s))\| ds \right)$$
$$\le C\delta\varepsilon + C\|m\|_X^{1+\alpha} \int_0^\infty e^{-t\sigma} ds \le C\delta\varepsilon + \frac{C}{\sigma}\|m\|_X^{1+\alpha} \le \varepsilon/2,$$

as long as $\delta : C\delta < 1/4$ and $\varepsilon : \frac{C}{\sigma}\varepsilon^\alpha < 1/4$.

To estimate $M_1(\tilde{m})$, we have

$$\max_{1\le j\le M} |\tilde{a}_j(t)| \le \int_t^\infty \|F(m(s))\| ds \le C\|m\|_X^{1+\alpha} \times \int_t^\infty e^{-s\sigma} ds \le \frac{C}{\sigma}\|m\|_X^{1+\alpha} e^{-t\sigma},$$

whence

$$M_1(\tilde{m}) \le C\varepsilon^{1+\alpha} \le \varepsilon,$$

provided $C\varepsilon^\alpha < 1$. Next,

$$\max_{1\le j\le d} |\tilde{b}_j(t)| \le e^{-t}|\tilde{y}^\infty| + \max_{1\le j\le d} \frac{1}{\|\partial_j\phi\|}\int_t^\infty \|F(m(s))\| ds$$
$$\le e^{-t}M_0(\tilde{m}) + \frac{C}{\sigma}\|m\|_X^{1+\alpha} e^{-t\sigma}.$$

Thus, taking into account that we have already established $M_0(\tilde{m}) \le \varepsilon/2$, and $\sigma < 1$,

$$M_2(\tilde{m}) = \sup_{0\le t<\infty} e^{t\sigma} \max_{1\le j\le d} |b_j(t)| \le M_0(\tilde{m}) + \frac{C}{\sigma}\|m\|_X^{1+\alpha} \le \varepsilon,$$

if $\frac{C}{\sigma}\varepsilon^\alpha < 1/4$.

Regarding $M_3(\tilde{m})$, we have

$$\max_{1\le j\le N} |\tilde{c}_j(t)| \le e^{-t\mu_{\min}^2}\|f\|_{L^2} + \int_0^t e^{(s-t)\mu_{\min}^2}\|F(m(s))\| ds$$

where $\mu_{\min}^2 := \min(\mu_1^2, \dots, \mu_N^2) > \sigma$. Hence,

$$\max_{1\le j\le N} |\tilde{c}_j(t)| \le \delta\varepsilon e^{-t\mu_{\min}^2} + \|m\|_X^{1+\alpha}\int_0^t e^{(s-t)\mu_{\min}^2} e^{-s\sigma} ds$$
$$\le \delta\varepsilon e^{-t\sigma} + C\|m\|_X^{1+\alpha} e^{-t\sigma}.$$

Thus

$$M_3(\tilde{x}) \le \varepsilon,$$

so long as $\delta < 1/2$ and $C\varepsilon^\alpha < 1/2$.

To finish off the argument, we need to establish an appropriate estimate for $M_4(\tilde{m})$. We have the following estimate from functional calculus for the self-adjoint operator $\mathcal{H}$

$$\|e^{-t\mathcal{H}}P_{a.c.}(\mathcal{H})\|_{L^2\to L^2} = \sup_{\lambda\in\sigma_{a.c.}(\mathcal{H})=[1,\infty)} e^{-t\lambda} = e^{-t}.$$

In order to establish the corresponding bounds in Sobolev spaces (and in particular in $\dot{W}^{1,q_0}$ as required here), we shall need to use the machinery of wave operators to translate various classical estimates for the heat semigroup $e^{t\Delta}$ to estimates for $e^{-t\mathcal{H}}P_{a.c.}(\mathcal{H})$.

### 2.3. Wave operators for Schrödinger operators and consequences

We discuss the boundedness of the wave operator $W$, associated to a Schrödinger operator in the form $H_V = -\Delta + V$, where $V$ is a real-valued potential. Define

$$\mathcal{W}_\pm := s - \lim_{t\to\pm\infty} e^{itH_V} e^{it\Delta}$$

where the limit is understood in the strong operator topology on $B(L^2(\mathbf{R}^d))$. While the $L^2$ boundedness of $W$ is well-understood and relatively easy to deduce, the $L^p \to L^p, 1 < p < \infty$ and more generally $W^{s,p}$ bounds are highly nontrivial. Nevertheless, in the last fifteen years a coherent theory of these estimates has emerged (together with far reaching applications). We summarize the results in the following

**Lemma 1.** *Let $V : \mathbf{R}^d \to \mathbf{R}^1$, $d \ge 1$ be a potential, which is $C^2$ and decaying with its derivatives. Assume also that the operator $H_V$ has neither resonance nor an eigenvalue at zero*[6]*. Then, for every $s, p : 1 \ge s \ge 0, 1 < p < \infty$, the wave operators $\mathcal{W}_\pm, \mathcal{W}_\pm^*$ are bounded operators from $W^{s,p}$ to itself.*

Lemma 1 is shown (under less restrictive assumptions on $V$) by Weder, [18] for $d = 1$, by Yajima, [19] for the case $d = 2$, in [20] for $d \ge 3$ and $d$ odd and in [21], $d \ge 4$ and even.

What is the the relevance of such a result to our problem? In our case, the relevant potential $V = -\mathcal{N}'(\phi)$ satisfies the assumptions of Lemma 1. Thus, one may write (see for instance formula (1.8) in [20])

$$f(H_V)P_{a.c.}(H_V) = \mathcal{W}_\pm f(H_0)\mathcal{W}_\pm^* \tag{21}$$

for any Borel function $f$. An easy computation shows that

$$\mathcal{W}_\pm = \lim_{t\to\pm\infty} e^{itH_V} e^{it\Delta} = \lim_{t\to\pm\infty} e^{it(-\Delta+1+V)} e^{-it(-\Delta+1)}$$

and hence

$$e^{-t\mathcal{H}}P_{a.c.}(\mathcal{H}) = \mathcal{W}_\pm e^{-t(-\Delta+1)}\mathcal{W}_\pm^*.$$

It is easy to see now, based on the last formula

$$\|e^{-t\mathcal{H}}P_{a.c.}(\mathcal{H})\|_{B(W^{s,q})} = \|\mathcal{W}_\pm e^{-t(-\Delta+1)}\mathcal{W}_\pm^*\|_{W^{s,q}}$$
$$\le \|\mathcal{W}_\pm\|_{B(W^{s,q})}\|\mathcal{W}_\pm^*\|_{B(W^{s,q})}\|e^{-t(-\Delta+1)}\|_{B(W^{s,q})} \le C_{q,s}e^{-t}.$$

Hence

$$\|\tilde{z}(t)\|_{L^2\cap\dot{W}^{1,q_0}} \le \|e^{-t\mathcal{H}}P_{a.c.}(\mathcal{H})f\|_{L^2\cap\dot{W}^{1,q_0}} + \int_0^t \|e^{-(t-s)\mathcal{H}}P_{a.c.}(\mathcal{H})F(m(s))\|_{L^2\cap\dot{W}^{1,q_0}} ds$$
$$\le e^{-t}\|f\|_{L^2\cap\dot{W}^{1,q_0}} + C\|m\|_X^{1+\alpha}\int_0^t e^{(s-t)}e^{-s\sigma} ds \le e^{-t}\delta\varepsilon + C\varepsilon^{1+\alpha}e^{-t\sigma}$$

and hence $M_4(\tilde{m}) \le \delta\varepsilon + C\varepsilon^{1+\alpha} \le \varepsilon$, provided $\delta < 1/2; C\varepsilon^\alpha < 1/2$.

Thus, we have shown that the mapping $\Lambda$ sends the ball $B(X, \varepsilon)$ into itself. Moreover, if one uses the more general Hölder continuity estimate, rather than (11) in the arguments above, it is easy to see that

$$\|\Lambda(m) - \Lambda(n)\|_X \le C\|m - n\|_X(\|m\|_X^\alpha + \|n\|_X^\alpha), \tag{22}$$

which provides that the mapping $\Lambda : B(X, \varepsilon) \to B(X, \varepsilon)$ is indeed a contraction, whose fixed point is the desired solution. Note

[6] In the case $d = 1$, this requirement can be dropped, see [18], but not in the case $d \ge 3$.

that by definition the defining functionals of the stable manifold $h_1, \dots, h_M$ are (nonlinearly) defined through (see (17))

$$h_j(f) := a_j(0) = \int_0^\infty e^{-(s-t)\sigma_j^2} \langle F(m(s)), \psi_j \rangle ds.$$

It follows from (20) that

$$|h_j(f)| \le \int_0^\infty e^{-s\sigma_j^2} \|F(m(s))\| ds \le C\|m\|_X^{1+\alpha} \times \int_0^\infty e^{-s\sigma_j^2} e^{-s\sigma(1+\alpha)} ds \le C\varepsilon^{1+\alpha}.$$

Regarding the approximate formula for $y^\infty$, we have from (18) that

$$y_j^\infty = \frac{1}{\|\partial_j \phi\|_{L^2}^2} \langle f, \partial_j \phi \rangle + O(\varepsilon^{1+\alpha}),$$

since $|y_j^\infty - \|\partial_j\phi\|_{L^2}^{-2} \langle f, \partial_j \phi\rangle|$ is estimated by $\int_0^\infty \|F(m(s))\| ds \le C\varepsilon^{1+\alpha} \int_0^\infty e^{-s\sigma} ds < C\varepsilon^{1+\alpha}$.

## 3. Proof of Theorem 2: Preliminaries

The argument is in many ways similar to the proof of Theorem 1. Let us mention, though, that one main difference is that by choosing an even initial data, we are in essence destroying the eigenvalue at zero. More precisely, since the evolution preserves even solutions and the zero eigenvalue has only odd eigenfunctions, the whole evolution proceeds perpendicularly to that marginally stable direction and the ansatz (6) does not contain the asymptotic phase function $y(t)$ and $\sum_j b_j \partial_j \psi$.

Next, we derive the equations satisfied by the unknowns $a(t)$, $\mathbf{z}(t)$. We have by (6)

$$\mathbf{z}_{tt} + \mathcal{H} z + \psi(a''(t) - \sigma^2 a(t)) - F(t, x) = 0,$$
$$F(t, x) = \mathcal{N}(\phi + a(t)\psi + \mathbf{z}(t)) - \mathcal{N}(\phi) - \mathcal{N}'(\phi)(a(t)\psi + \mathbf{z}(t)) \tag{23}$$

where $\mathcal{N}(u) = |u|^{p-1}u$. From here, taking the spectral projections, we derive the equations

$$a''(t) - \sigma^2 a(t) - \langle F(t, \cdot), \psi \rangle = 0 \tag{24}$$
$$\mathbf{z}_{tt} + \mathcal{H} z - P_{a.c.}[F] = 0. \tag{25}$$

### 3.1. Analysis of the $a(t)$ equation

We have the explicit solution of (24) in the form

$$a(t) = \cosh(\sigma t)a(0) + \frac{1}{\sigma} \sinh(\sigma t) a'(0) + \frac{1}{\sigma} \int_0^t \sinh(\sigma(t-s)) \langle F(s, \cdot), \psi \rangle ds. \tag{26}$$

Our goal is to achieve a vanishing solution. The first step is to ensure that $\lim_{t\to\infty} a(t) = 0$. There is one term in (26) that vanishes whenever $\langle F(t, \cdot), \psi \rangle \to 0$. Namely, we will show

$$\lim_{t\to\infty} e^{-t\sigma} \int_0^t e^{s\sigma} \langle F(s, \cdot), \psi \rangle ds = 0.$$

Indeed, we estimate as follows

$$e^{-t\sigma} \int_0^t e^{s\sigma} |\langle F(s, \cdot), \psi \rangle| ds \le e^{-t\sigma} \sup_s |\langle F(s, \cdot), \psi \rangle| \int_0^{t/2} e^{s\sigma} ds + e^{-t\sigma} \sup_{s \ge t/2} |\langle F(s, \cdot), \psi \rangle| \int_{t/2}^t e^{s\sigma} ds.$$

Clearly, the last expression converges to zero, as long as $\lim_{t\to\infty} |\langle F(t, \cdot), \psi \rangle| = 0$.

We now need to make sure that the remaining terms also vanish as $t \to \infty$. We group them as follows

$$\frac{e^{t\sigma}}{2} \left[ a(0) + \frac{a'(0)}{\sigma} + \frac{1}{\sigma} \int_0^t e^{-\sigma s} \langle F(s, \cdot), \psi \rangle ds \right].$$

It is now clear that in order to achieve the vanishing of these terms as $t \to \infty$, we shall need to enforce the nonlinear relation

$$a(0) + \frac{a'(0)}{\sigma} + \frac{1}{\sigma} \int_0^\infty e^{-\sigma s} \langle F(s, \cdot), \psi \rangle ds = 0. \tag{27}$$

Note that due to the exponential factor $e^{t\sigma}$ in front, this is the only hope to have $a(t) \to 0$. On the other hand, if (27) holds, we have by the L'Hospital's rule,

$$\lim_{t\to\infty} \frac{a(0) + \frac{a'(0)}{\sigma} + \frac{1}{\sigma} \int_0^t e^{-\sigma s} \langle F(s, \cdot), \psi \rangle ds}{e^{-t\sigma}} = \frac{1}{\sigma} \lim_{t\to\infty} \frac{e^{-t\sigma} \langle F(t, \cdot), \psi \rangle}{-\sigma e^{-t\sigma}} = 0,$$

provided $\langle F(t, \cdot), \psi \rangle \to 0$. In terms of the initial data we have

$a(0) = \langle f_1, \psi \rangle + h(f_1, f_2)$; $a'(0) = \langle f_2, \psi \rangle$

and taking into account $\langle f_1 + \frac{1}{\sigma} f_2, \psi \rangle = 0$, we see that the validity of (27) is equivalent to

$$h(f_1, f_2) = -\frac{1}{\sigma} \int_0^\infty e^{-\sigma s} \langle F(s, \cdot), \psi \rangle ds. \tag{28}$$

Thus, the solution of the $a$ equation, which obeys $a(t) \to 0$ will take the form

$$a(t) = \frac{e^{-t\sigma}}{2} \left[ a(0) - \frac{1}{\sigma} a'(0) \right] - \frac{1}{2\sigma} \int_0^t e^{-\sigma(t-s)} \langle F(s, \cdot), \psi \rangle ds - \frac{1}{2\sigma} \int_t^\infty e^{\sigma(t-s)} \langle F(s, \cdot), \psi \rangle ds. \tag{29}$$

In the derivation of the last formula, we have explicitly assumed the validity of (27), which still needs to be enforced nonlinearly in the evolution.

### 3.2. Linear estimates for the Klein–Gordon equation

Regarding the Klein–Gordon equation describing the evolution of the $\mathbf{z}$ variable, we write its solution as follows

$$\mathbf{z}(t) = \cos(t\sqrt{\mathcal{H}}) P_{a.c.}(\mathcal{H})[f_1] + \frac{\sin(t\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}} P_{a.c.}(\mathcal{H})[f_2] + \int_0^t \frac{\sin((t-s)\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}} P_{a.c.}(\mathcal{H})[F(s, \cdot)] ds, \tag{30}$$

where we have taken into account $P_{a.c.}(\mathcal{H})\psi = 0$.

The next goal is to analyze this equation for $\mathbf{z}$ for small data in appropriate function spaces. The necessary tools to do that have been developed recently by a number of authors — in particular we refer to the so-called Strichartz estimates. These are readily available only for the evolution of the "free" Klein–Gordon equation, i.e. generated by $e^{it\sqrt{\mathcal{H}_0}}$, where $\mathcal{H}_0 := -\Delta + 1$. In addition, we use the wave operators to obtain Strichartz estimates for the perturbed evolution $e^{it\sqrt{\mathcal{H}}}$.

For the Strichartz estimates, we follow the recent paper [17] by Nakamura and Ozawa. We do not list the full statements in [17], as they have too many parameters. Instead, we select the value of the parameters, which are most advantageous for our nonlinear problem and state those explicitly. We need a definition first.

**Definition 1.** We say that a pair $(q,r)$ is KG admissible (sharp KG admissible respectively), if $q,r\geq 2: 2/q+d/r\leq d/2$ ($q,r\geq 2: 2/q+d/r=d/2$ respectively) and $(q,r,d)\neq(2,\infty,2)$.

**Lemma 2** (*see Lemma 2.1 in [17] with $\sigma=d$, $\lambda=(d+2)/2$*). *Let $(q,r)$, $(q_1,r_1)$ be both KG admissible pairs and $s\geq 0$. Then*

$$\|e^{it\sqrt{\mathcal{H}_0}}f\|_{L^q_t W^{s,r}_x}\leq C\|f\|_{H^{s+\frac{d+2}{2}(\frac12-\frac1r)}},$$

$$\left\|\int_0^t \frac{\sin((t-\tau)\sqrt{\mathcal{H}_0})}{\sqrt{\mathcal{H}_0}}G(\tau,\cdot)\mathrm{d}\tau\right\|_{L^q_t W^{s,r}_x}\leq C\|G\|_{L^{q_1'}_t W^{s-1+\frac{d+2}{2}(\frac{1}{r_1'}-\frac1r),r_1'}}.$$

Based on (21) with $f(\xi)=e^{it\sqrt{\xi}}$, we may now translate Lemma 2 in terms of $\mathcal{H}$ as follows.

**Lemma 3.** *For all KG admissible pairs $(q,r)$, $(\tilde q,\tilde r)$, where $r,\tilde r<\infty$ and $s\geq 0$, there exists $C=C(s,d)$, so that*

$$\|e^{it\sqrt{\mathcal{H}}}P_{a.c.}(\mathcal{H})f\|_{L^q_t W^{s,r}_x}\leq C\|f\|_{H^{s+\frac{d+2}{2}(\frac12-\frac1r)}},\tag{31}$$

$$\left\|\int_0^t \frac{\sin((t-\tau)\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}}P_{a.c.}(H_V)[G(\tau,\cdot)]\mathrm{d}\tau\right\|_{L^q_t W^{s,r}_x}\leq C\|G\|_{L^{\tilde q'}_t W^{s-1+\frac{d+2}{2}(\frac{1}{\tilde r'}-\frac1r),\tilde r'}}.\tag{32}$$

**Proof.** For (31), we have

$$\begin{aligned}\|e^{it\sqrt{\mathcal{H}}}P_{a.c.}(\mathcal{H})f\|_{L^q_t W^{r,s}_x}&=\|\mathcal{W}_\pm e^{it\sqrt{-\Delta+1}}\mathcal{W}^*_\pm f\|_{L^q_t W^{r,s}_x}\\&\leq\|\mathcal{W}_\pm\|_{B(W^{r,s}_x)}\|e^{it\sqrt{-\Delta+1}}\mathcal{W}^*_\pm f\|_{L^q_t W^{r,s}_x}.\end{aligned}$$

Applying the Strichartz estimates for the free Klein–Gordon (Lemma 2) yields

$$\begin{aligned}\|e^{it\sqrt{-\Delta+1}}\mathcal{W}^*_\pm f\|_{L^q_t W^{r,s}_x}&\leq C\|\mathcal{W}^*_\pm f\|_{H^{s+\frac{d+2}{2}\left(\frac12-\frac1r\right)}}\\&\leq C\|\mathcal{W}^*_\pm\|_{B\left(H^{s+\frac{d+2}{2}\left(\frac12-\frac1r\right)}\right)}\|f\|_{H^{s+\frac{d+2}{2}\left(\frac12-\frac1r\right)}}.\end{aligned}$$

Composing all the estimates above implies the statement with a constant depending on the Strichartz constants times operator norms of $\mathcal{W}_\pm$ in various Sobolev spaces. The treatment of (32) is similar and it is therefore omitted. □

## 4. Proof of Theorem 2: Main argument

We begin the proof of Theorem 2 with an elementary proposition regarding a pointwise estimate for the non-linearity $F$.

**Proposition 4.** *Let $p\geq 2$ and*

$$G(h)=|\phi+h|^{p-1}(\phi+h)-\phi^p-p\phi^{p-1}h.$$

*Then, one has the pointwise estimate*

$$|G(h)|\leq C_p(\phi^{p-2}|h|^2+|h|^p)$$

*As a corollary, one obtains the following estimate for the nonlinearity $F=G(a(t)\psi+\mathbf{z}(t,\cdot))$ (as defined in (23)),*

$$\begin{aligned}|F(t,x)|&\leq C_p(\phi^{p-2}(|a(t)|^2\psi^2+|\mathbf{z}(t)|^2)\\&\quad+|a(t)|^p\psi^p+|\mathbf{z}(t)|^p).\end{aligned}\tag{33}$$

*Also, if $p\geq 3$, one has the following estimate for the derivative*

$$|\nabla G(h)|\leq C_p(\phi^{p-3}|h|^2|\nabla\phi|+|\nabla h||h|\phi^{p-2}+|\nabla h||h|^{p-1}),$$

*whence*

$$\begin{aligned}|\nabla F(t,x)|&\leq H_1(t,x)(|a(t)|^2+|a(t)|^p+|\mathbf{z}(t,x)|^2+|\mathbf{z}(t,x)|^p)\\&\quad+H_2(t,x)|\nabla\mathbf{z}||a(t)|^{p-1}+|\nabla\mathbf{z}||\mathbf{z}|^{p-1},\end{aligned}\tag{34}$$

*where $H_i=H_i(\phi,\psi)$, $i=1,2\in L^1\cap L^\infty$.*

**Proof.** We only show (33), since the proof of (34) is similar. Split into the cases $|h|>\phi/100$ and otherwise. Split into the cases $\{x:|h(x)|>\phi(x)/100\}$ and otherwise. If $|h(x)|>\phi(x)/100$, we take absolute values and estimate each term separately by $C_p|h(x)|^p$. On the other hand, if $|h(x)|\ll\phi(x)$, we have $\phi+h>0$ and hence

$$\begin{aligned}|(\phi+h)^p-\phi^p-p\phi^{p-1}h|&=\phi^p\left|\left(1+\frac{h}{\phi}\right)^p-1-p\frac{h}{\phi}\right|\\&\leq C_p\phi^p\frac{h^2}{\phi^2}=C_p\phi^{p-2}|h|^2.\end{aligned}$$

Combining the estimates in both cases yields the result. □

### 4.1. Setting the contraction map and the function spaces

We have shown that one may reduce the problem to the following integral equation for the unknowns $m(t):=(h,a(t),\mathbf{z}(t))$, defined by (28)–(30). That is, for a nonlinearity $F=F(m)$ (explicitly defined in (23)), define $\tilde m=\Lambda(m)$ via

$$\begin{aligned}\tilde h&=-\frac{1}{\sigma}\int_0^\infty e^{-\sigma s}\langle F(m(s)),\psi\rangle\mathrm{d}s,\\ \tilde a(t)&=\frac{e^{-t\sigma}}{2}[2\langle f_1,\psi\rangle+\tilde h]-\frac{1}{2\sigma}\int_0^t e^{-\sigma(t-s)}\langle F(m(s)),\psi\rangle\mathrm{d}s\\&\quad-\frac{1}{2\sigma}\int_t^\infty e^{\sigma(t-s)}\langle F(m(s)),\psi\rangle\mathrm{d}s,\\ \tilde{\mathbf{z}}(t)&=\cos(t\sqrt{\mathcal{H}})P_{a.c.}(\mathcal{H})[f_1]+\frac{\sin(t\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}}P_{a.c.}(\mathcal{H})[f_2]\\&\quad+\int_0^t\frac{\sin((t-s)\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}}P_{a.c.}(\mathcal{H})[F(m(s))]\mathrm{d}s.\end{aligned}$$

Note in the definition of $\tilde a$ above, we have used the formula $2\langle f_1,\psi\rangle+\tilde h$ instead of $a(0)-\frac{a'(0)}{\sigma}$. These two expressions coincide, whenever $h=\tilde h$ and $\langle f_1+\frac{f_2}{\sigma},\psi\rangle=0$.

Our goal is to show that such a map is a contraction in an appropriate metric space, which must in turn guarantee $a(t)\to 0$ and $\mathbf{z}(t)\to 0$.

Fix $q_0,r_0: 2<q_0<8/3; 1/q_0+1/r_0=1/2$. Introduce the norms

$$M_0(m):=|h|;$$

$$M_1(m):=\begin{cases}\|a\|_{L^1_t([0,\infty))\cap L^\infty_t([0,\infty))} & d=3,4\\ \|a\|_{L^2_t([0,\infty))\cap L^\infty_t([0,\infty))} & d=2\end{cases}$$

$$M_2(m)=\begin{cases}\|\mathbf{z}\|_{L^\infty_t H^2_x(\mathbf{R}^d)\cap L^2_t W^{3/2-1/d,2d/(d-2)}} & d=3,4\\ \|\mathbf{z}\|_{L^\infty_t H^1_x(\mathbf{R}^2)\cap L^{q_0}_t W^{1-2/q_0,r_0}}, & d=2.\end{cases}$$

The Banach space $X$ is now defined as the set of all $m=(h,a(t),\mathbf{z}(t))$ with a norm $\|m\|_X:=\max(M_0(m),M_1(m),M_2(m))$. Note that by Sobolev embedding and Gagliardo–Nirenberg's inequality, we have for every KG admissible pair $(q,r)$

$$\|\mathbf{z}\|_{L^q_t W^{2-(d/2-2/q-d/r)-1/q-2/(dq),r}_x}\leq M_2(m).\tag{35}$$

In addition, we will also use Gagliardo–Nirenberg's inequality (or log-convexity of $L^r$ norms), namely for $w\geq 2$,

$$\|a\|_{L^w(0,\infty)}\leq M_1(m).$$

For $d=3,4$, this follows from

$$\|a\|_{L^w(0,\infty)}\leq\|a\|^{1/w}_{L^1(0,\infty)}\|a\|^{1-1/w}_{L^\infty(0,\infty)}\leq M_1(m)$$

and in similar way for $d=2$.

Now that we have set the stage, we proceed to show that there exists $0 < \varepsilon = \varepsilon(d,p) \ll 1$ and $\delta = \delta(d,p)$, so that whenever $\|(f_1,f_2)\|_{H^2\times H^1} < \delta\varepsilon$, we have that $\Lambda : B_X(\varepsilon) \to B_X(\varepsilon)$ is a contraction.

Thus, assume that $\|m\|_X \le \varepsilon$. We have

$$M_0(m) = |\tilde{h}| \le \frac{1}{\sigma}\int_0^\infty e^{-\sigma s}|\langle F(m(s)),\psi\rangle|ds.$$

From (33), we estimate

$$|\langle F(m(s)),\psi\rangle| \le C(|a(s)|^2 + \|\mathbf{z}(s,\cdot)\|^2_{L^2_x} + |a(t)|^p + \|\mathbf{z}(s,\cdot)\|^p_{L^p_x}), \qquad (36)$$

where $C$ depends upon various $L^w$ norms of the decaying functions $\phi$, $\psi$. It follows that

$$\begin{aligned}M_0(m) &\le C\int_0^\infty e^{-\sigma s}(|a(s)|^2 + \|\mathbf{z}(s,\cdot)\|^2_{L^2_x} + |a(t)|^p + \|\mathbf{z}(s,\cdot)\|^p_{L^p_x})ds\\ &\le C(\|a\|^2_{L^2} + \|\mathbf{z}\|^2_{L^\infty_t L^2_x} + \|a\|^p_{L^p} + \|\mathbf{z}\|^p_{L^\infty_t L^p_x})\\ &\le C(M_1(m)^2 + M_2(m)^2 + M_1(m)^p + M_2(m)^p)\\ &\le C(\varepsilon^2+\varepsilon^p) \le \frac{\varepsilon}{10}\min(1,\sigma),\end{aligned}$$

provided $C(\varepsilon + \varepsilon^{p-1}) \le 1/10$. Here, we have used that $(\infty, p)$ is a KG admissible pair and hence $\|\mathbf{z}\|_{L^\infty_t L^p_x} \le M_2(m)$.

### 4.2. Estimating $M_1(\tilde{m})$

The quantity $M_1(\tilde{m})$ has two components. Firstly, we estimate

$$\begin{aligned}\sup_t |\tilde{a}(t)| \le{}& \frac{1}{2}(2|\langle f_1,\psi\rangle| + |\tilde{h}|)\\ &+ \frac{1}{2\sigma}\sup_t \int_0^t e^{-\sigma(t-s)}|\langle F(m(s)),\psi\rangle|ds\\ &+ \frac{1}{2\sigma}\sup_t \int_t^\infty e^{\sigma(t-s)}|\langle F(m(s)),\psi\rangle|ds.\end{aligned}$$

From (36) and the estimates for $M_0(\tilde{m})$, it follows

$$\begin{aligned}\sup_t |\tilde{a}(t)| &\le \delta\varepsilon + \frac{\varepsilon}{10} + \frac{1}{\sigma^2}\sup_s |\langle F(m(s)),\psi\rangle|\\ &\le \delta\varepsilon + \varepsilon/10 + \frac{C}{\sigma^2}(M_1(m)^2 + M_2(m)^2\\ &\quad + M_1(m)^p + M_2(m)^p) \le \varepsilon,\end{aligned}$$

provided $\delta < 1/2$ and $C(2\varepsilon + 2\varepsilon^{p-1}) \le \sigma^2/4$. This takes care of the term in $M_1(\tilde{m})$ that contains $\|\tilde{a}\|_{L^\infty}$. We now need to show estimates on the integrability of $\tilde{a}$. This depends on the dimension.

For $d = 3, 4$, we have by Hausdorf–Young's inequality

$$\begin{aligned}\|\tilde{a}\|_{L^1_t} &\le (\|f_1\|_{L^2} + |\tilde{h}|)\int_0^\infty e^{-t\sigma}dt\\ &\quad + \frac{1}{2\sigma}\|e^{-\sigma|\cdot|}\|_{L^1}\|\langle F(m(s)),\psi\rangle\|_{L^1_s}\\ &\le \frac{1}{\min(1,\sigma)}\left(\delta\varepsilon + \frac{\varepsilon}{10}\min(1,\sigma)\right)\\ &\quad + \frac{1}{2\sigma^2}\int_0^\infty |\langle F(m(s)),\psi\rangle|ds.\end{aligned}$$

We need to apply (33) more judiciously, in order to fit the setup for the contraction argument. Namely, since $p < 1 + 4/(d-2) < 2d/(d-2)$, we bound By Hölder's inequality

$$|\langle F(m(s)),\psi\rangle| \le C(|a(s)|^2 + \|\mathbf{z}(s,\cdot)\|^2_{L^{2d/(d-2)}_x} + |a(s)|^p + \|\mathbf{z}(s,\cdot)\|^p_{L^{2d/(d-2)}_x}).$$

It follows that

$$\int_0^\infty |\langle F(m(s)),\psi\rangle|ds \le C(\|a\|^2_{L^2_t} + \|a\|^p_{L^p_t} + \|\mathbf{z}\|^2_{L^2_t L^{2d/(d-2)}_x} + \|\mathbf{z}\|^p_{L^p_t L^{2d/(d-2)}_x}).$$

It remains to observe that since $(2, 2d/(d-2))$, $(p, 2d/(d-2))$ are KG admissible

$$\begin{aligned}&\|a\|_{L^2_t}, \|a\|_{L^p_t} \le M_1(m) \le \varepsilon\\ &\|\mathbf{z}\|_{L^2_t L^{2d/(d-2)}_x}, \|\mathbf{z}\|_{L^p_t L^{2d/(d-2)}_x} \le M_2(m) \le \varepsilon.\end{aligned}$$

All in all, we have shown that

$$\|\tilde{a}\|_{L^1_t} \le \frac{1}{\min(1,\sigma)}\left(\delta\varepsilon + \frac{\varepsilon}{10}\min(1,\sigma)\right) + C_\sigma(2\varepsilon^2 + 2\varepsilon^p),$$

and hence it suffices to require $\delta < \min(1,\sigma)/2$ and $C_\sigma(2\varepsilon + 2\varepsilon^{p-1}) \le \varepsilon/4$ in order to conclude that

$$M_1(\tilde{m}) = \max(\|\tilde{a}\|_{L^\infty_t}, \|\tilde{a}\|_{L^1_t}) \le \varepsilon.$$

For $d = 2$, we use the bound (33) to derive

$$|\langle F(m(s)),\psi\rangle| \le C(|a(s)|^2 + |a(s)|^p + \|\mathbf{z}(s,\cdot)\|^2_{L^4} + \|\mathbf{z}(s,\cdot)\|^p_{L^p}),$$

whence

$$\begin{aligned}\|\tilde{a}\|_{L^2_t} &\le (\|f_1\|_{L^2} + |\tilde{h}|)\left(\int_0^\infty e^{-2t\sigma}dt\right)^{1/2}\\ &\quad + \frac{1}{2\sigma}\|e^{-\sigma|\cdot|}\|_{L^1}\|\langle F(m(s)),\psi\rangle\|_{L^2_s}\\ &\le \frac{1}{\min(1,\sigma)}\left(\delta\varepsilon + \frac{\varepsilon}{10}\min(1,\sigma)\right) + C(\|a\|^2_{L^4}\\ &\quad + \|a\|^p_{L^{2p}} + \|\mathbf{z}\|^2_{L^4_t L^4} + \|\mathbf{z}\|^p_{L^{2p}_t L^p}).\end{aligned}$$

Now, since $(4, 4)$ and $(2p, p)$ are KG admissible[7],

$$\begin{aligned}&\|a\|_{L^4_t}, \|a\|_{L^{2p}_t} \le M_1(m) \le \varepsilon\\ &\|\mathbf{z}\|_{L^4_t L^4}, \|\mathbf{z}\|_{L^{2p}_t L^p} \le M_2(m) \le \varepsilon.\end{aligned}$$

This implies, by requiring $\delta < \min(1,\sigma)/2$ and $C_\sigma(2\varepsilon + 2\varepsilon^{p-1}) \le \varepsilon/4$ that $M_1(\tilde{m}) \le \varepsilon$ in the case $d = 2$ as well.

### 4.3. Estimates on $M_2(\tilde{m})$

The estimates for $\mathbf{z}$ go through the Strichartz estimates of Lemma 3, more precisely, (31) and (32).

In the case $d = 3, 4$, we have

$$\begin{aligned}&\|\tilde{\mathbf{z}}\|_{L^\infty_t H^2 \cap L^2 W^{3/2-1/d,2d/(d-2)}_x}\\ &\le C\|\cos(t\sqrt{\mathcal{H}})P_{a.c}f_1\|_{L^\infty_t H^2\cap L^2 W^{3/2-1/d,2d/(d-2)}_x}\\ &+ \left\|\frac{\sin(t\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}}P_{a.c}f_2\right\|_{L^\infty_t H^2\cap L^2 W^{3/2-1/d,2d/(d-2)}_x}\\ &+ \left\|\int_0^t \frac{\sin((t-s)\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}}P_{a.c.}(\mathcal{H})\right.\\ &\times \left.[F(m(s))]ds\right\|_{L^\infty_t H^2\cap L^2 W^{3/2-1/d,2d/(d-2)}_x,}\\ &\le C(\|(f_1,f_2)\|_{H^2(\mathbf{R}^d)\times H^1(\mathbf{R}^d)} + \|F\|_{L^1_t H^1_x}).\end{aligned}$$

[7] Here we have used $p \ge 1 + 4/d = 3$ for $d = 2$.

By (33) and (34), we see that

$$\|F\|_{L^1_tL^2_x} + \|\nabla F\|_{L^1_tL^2_x} \le C(\|a\|^2_{L^2_t} + \|a\|^p_{L^p_t} + \|\mathbf{z}\|^2_{L^2_tL^{2d/(d-2)}_x}$$
$$+ \|\mathbf{z}\|^p_{L^p_tL^{2p}_x}) + C(\|a\|^{p-1}_{L^{p-1}_t}\|\nabla z\|_{L^\infty L^2} + \|\nabla \mathbf{z}\|_{L^2L^{2d/(d-2)}_x}\|z\|^{p-1}_{L^{2(p-1)}_tL^{(p-1)d}_x}).$$

Again, our old arguments apply (recall $p \ge 2$) to see that

$$\|a\|_{L^2_t}, \|a\|_{L^{p-1}_t}, \|a\|_{L^p_t} \le M_1(m) \le \varepsilon$$

and

$$\|\mathbf{z}\|_{L^2_tW^{1,2d/(d-2)}_x} \le M_2(m) \le \varepsilon.$$

*However, the norms* $\|\mathbf{z}\|_{L^p_tL^{2p}_x}$, $\|\mathbf{z}\|_{L^{2(p-1)}_tL^{(p-1)d}_x}$ *cannot be controlled by* $M_2(m)$, *unless* $(p, 2p)$, $(2(p-1), (p-1)d)$ *are both KG admissible pairs. Hence, we need to require the conditions*

$$\frac{2}{p} + \frac{d}{2p} \le \frac{d}{2}$$
$$\frac{2}{2(p-1)} + \frac{d}{(p-1)d} \le \frac{d}{2}$$

both of which are equivalent to $p \ge 1 + 4/d$. This is the only place where this requirement appears. On the other hand, this exponent naturally appears in the study of global solutions for Klein–Gordon equations with small data, [17]. Clearly, since our setup is more sophisticated than [17], it is expected that such a condition will be necessary. Thus,

$$\|\mathbf{z}\|_{L^p_tL^{2p}_x}, \|\mathbf{z}\|_{L^{2(p-1)}_tL^{(p-1)d}_x} \le M_2(m),$$

whence

$$\|\tilde{\mathbf{z}}\|_{L^\infty_tH^1\cap L^2W^{1/2-1/d,2d/(d-2)}_x} \le C(\delta\varepsilon + 2\varepsilon^2 + 2\varepsilon^p) \le \varepsilon,$$

if $C\delta \le 1/4$, $C(\varepsilon + \varepsilon^{p-1}) \le 1/4$.

In the case $d = 2$, we use (32) with exponents $\tilde{q} = \tilde{r} = 4$. We have

$$\|\tilde{\mathbf{z}}\|_{L^\infty_tH^1_x\cap L^{q_0}_tW^{r_0,1-2/q_0}}$$
$$\le C(\|(f_1, f_2)\|_{H^1(\mathbf{R}^2)\times L^2(\mathbf{R}^2)} + \|F\|_{L^{4/3}_tW^{1/2,4/3}(\mathbf{R}^2)}).$$

By Gagliardo–Nirenberg and the pointwise estimates (33) and (34), we have

$$\|F\|_{L^{4/3}_tW^{1/2,4/3}(\mathbf{R}^2)} \le \|F\|^{1/2}_{L^{4/3}_tL^{4/3}_x}\|\nabla F\|^{1/2}_{L^{4/3}_tL^{4/3}_x}$$
$$\le \frac{1}{2}(\|F\|_{L^{4/3}_tL^{4/3}_x} + \|\nabla F\|_{L^{4/3}_tL^{4/3}_x})$$
$$\le C(\|a\|^2_{L^{8/3}_t} + \|a\|^p_{L^{4p/3}_t} + \|\mathbf{z}\|^2_{L^{8/3}_tL^8} + \|\mathbf{z}\|^p_{L^{4p/3}_tL^{4p/3}_x})$$
$$+ \|\nabla \mathbf{z}\|_{L^\infty_tL^2_x}\|a\|_{L^{4(p-1)/3}} + \|\nabla \mathbf{z}\|_{L^\infty_tL^2_x}\|\mathbf{z}\|^{p-1}_{L^{4(p-1)/3}L^{4(p-1)}_x}.$$

Observe that the condition $p \ge 3 (= 1 + 4/d)$ implies that $4p/3 > 4(p-1)/3 \in (2, \infty)$ and $(4p/3, 4p/3)$, $(4(p-1)/3, 4(p-1))$ are KG admissible pairs (in 2 spatial dimensions) and hence

$$\|a\|_{L^{8/3}_t}, \|a\|_{L^{4p/3}_t}, \|a\|_{L^{4p/3}_t} \le M_1(m);$$
$$\|\nabla \mathbf{z}\|_{L^\infty_tL^2_x}, \|\mathbf{z}\|_{L^{4p/3}_tL^{4p/3}_x}, \|\mathbf{z}\|_{L^{4(p-1)/3}L^{4(p-1)}_x} \le M_2(m);$$

All in all,

$$\|\tilde{\mathbf{z}}\|_{L^\infty_tH^1_x\cap L^{q_0}_tW^{1-2/q_0,r_0}} \le C(\delta\varepsilon + \varepsilon^2 + \varepsilon^p),$$

whence $M_2(\tilde{m}) \le \varepsilon$, provided $C\delta < 1/4$, $C(\varepsilon + \varepsilon^{p-1}) \le 1/2$.

So far, we have established that $\Lambda : B_X(\varepsilon) \to B_X(\varepsilon)$ for appropriately chosen $\varepsilon$ and $\delta$, so that $\|(f_1, f_2)\|_{H^2\times H^1} \le \delta\varepsilon$ in $d = 2$ and $\|(f_1, f_2)\|_{H^1\times L^2} \le \delta\varepsilon$ in $d = 3, 4$ respectively. Regarding the contractivity of the map in the same space, it will suffice to run the same argument with all the estimates for $F(m)$ replaced by appropriate estimates for $F(m_1) - F(m_2)$. The situation is similar to the proof of (22), so we omit the details.

## Acknowledgments

Stanislavova supported in part by NSF-DMS 0807894. Stefanov supported in part by NSF-DMS 0908802.

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