DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KANSAS
SAMPLE MIDTERM MATH 800 - Spring 2020

Your Name: ______________________________

1	(50)	________
2	(50)	________
3	(50)	________
4	(50)	________
5	(50)	________
6	(50)	________
Total	(300)	________

(1) (50 points) Let $f \neq const.$ be an entire function, that is a holomorphic function in the whole complex plane. Assume that $L = \liminf_{z \to \infty} |f(z)| > 0$. Show that f vanishes somewhere in $\mathcal{C}$. This is an extension of the fundamental theorem of algebra.
Hint: Assume for a contradiction that f does not vanish and consider $g = 1/f$.

(2) (50 points) Let f be a holomorphic function in $D(0,1) \setminus \{0\}$. Prove that if 0 is an essential singularity for f, then it is an essential singularity for e^f as well.
Hint: Use Casorati-Weierstrass for f to rule out removable or pole options for e^f.

(3) (50 points) Let $f : \mathcal{C} \to \mathcal{C}$ is an entire function. Show that $f(\mathcal{C})$ is dense in $\mathcal{C}$. That is, for each $w \in \mathcal{C}$ and for each $\epsilon > 0$, there is z, so that $|f(z) - w| < \epsilon$.

(4) (50 points) Let $0 < a$. Show that there is no holomorphic function f in the punctured neighborhood $\{z : 0 < |z| < a\}$, so that $f^2(z) = z$.

(5) (50 points)
Let $f : D(0,1) \to \mathbb{C}$ be a holomorphic function on $D(0,1)$, with $\sup_{\xi:|\xi|\leq 1} |f(\xi)| \leq 1$. Show that for every $z : |z| < 1$, there is the bound

$$|f'(z)| \leq \frac{1}{1-|z|}.$$

Hint: Recall

$$f'(z) = \frac{1}{2\pi i} \oint_{|\xi - z| = r} \frac{f(\xi)}{(\xi - z)^2} d\xi,$$

whenever $D(z,r) \subset D(0,1)$.

(6) (50 points)
Let f be holomorphic in $D(0,1)$ and $|f(z)| \leq 1$. Show that $|f'(0)| \leq 1$.
Hint: Use the representation

$$f'(0) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(\xi)}{\xi^2} d\xi,$$

for any $0 < r < 1$, which follows from the Cauchy formula.