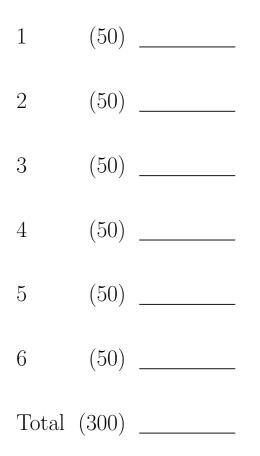
## DEPARTMENT OF MATHEMATICS UNIVERSITY OF KANSAS SAMPLE MIDTERM MATH 800 - Spring 2020

Your Name: \_\_\_\_\_



(1) (50 points) Let  $f \neq const$  be an entire function, that is a holomorphic function in the whole complex plane. Assume that  $L = \liminf_{z\to\infty} |f(z)| > 0$ . Show that f vanishes somewhere in C. This is an extension of the fundamental theorem of algebra. **Hint:** Assume for a contradiction that f does not vanish and consider

**Hint:** Assume for a contradiction that f does not vanish and consider g = 1/f.

(2) (50 points) Let f be a holomorphic function in  $D(0,1) \setminus \{0\}$ . Prove that if 0 is an essential singularity for f, then it is an essential singularity for  $e^f$  as well.

**Hint:** Use Casorati-Weierstrass for f to rule out removable or pole options for  $e^{f}$ .

- (3) (50 points) Let  $f : \mathcal{C} \to \mathcal{C}$  is an entire function. Show that  $f(\mathcal{C})$  is dense in  $\mathcal{C}$ . That is, for each  $w \in \mathcal{C}$  and for each  $\epsilon > 0$ , there is z, so that  $|f(z) - w| < \epsilon$ .
- (4) (50 points) Let 0 < a. Show that there is no holomorphic function f in the punctured neighborhood  $\{z : 0 < |z| < a\}$ , so that  $f^2(z) = z$ .
- (5) (50 points)

Let  $f: D(0,1) \to \mathbb{C}$  be a holomorphic function on D(0,1), with  $\sup_{\xi:|\xi|<1} |f(\xi)| \leq 1$ . Show that for every z: |z| < 1, there is the bound

$$|f'(z)| \le \frac{1}{1 - |z|}.$$

Hint: Recall

$$f'(z) = \frac{1}{2\pi i} \oint_{|\xi-z|=r} \frac{f(\xi)}{(\xi-z)^2} d\xi,$$

whenever  $D(z, r) \subset D(0, 1)$ .

(6) (50 points)

Let f be holomorphic in D(0,1) and  $|f(z)| \le 1$ . Show that  $|f'(0)| \le 1$ . **Hint:** Use the representation

$$f'(0) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(\xi)}{\xi^2} d\xi,$$

for any 0 < r < 1, which follows from the Cauchy formula.