MATH 800, Spring 2020

Daily Update:

January 22, Wednesday We began with the fundamental concepts in complex analysis - identifying the complex numbers with pairs of real numbers equipped with the algebraic operations of addition and multiplication and discussing various basic properties of complex numbers, polar representation, complex exponentials, Cauchy-Schwarz and triangle inequalities. We then spent some time working with complex polynomials and defining the partial differential operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$, which are linear, satisfy the Leibniz rule and play an important role in complex analysis.

January 24, Friday We defined holomorphic functions and proved that their real and imaginary parts satisfy the Cauchy-Riemann equations and are harmonic functions. We then worked on the question of the existence of a holomorphic function, whose real part is a given harmonic function and proved the result in the case of an open rectangle. We also discussed the existence of holomorphic antiderivative. This concludes chapter 1 and the assigned Homework set I is due on February 5.

Week 2, January 27-31 We spent this week discussing integrals in the complex plane setting and defined integral of a continuous function f over a smooth curve $\gamma(t) = \gamma_1(t) + i\gamma_2(t) : [a, b] \to U$ in an open set U as follows

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))(\gamma_1'(t) + i\gamma_2'(t))dt.$$

In the case of a **holomorphic function** $f : U \to \mathbf{C}$ we have the following

Proposition 1

$$\int_\gamma \frac{\partial f(z)}{\partial z} dz = f(\gamma(b)) - f(\gamma(a)).$$

Note that the condition that the function f is holomorphic is very important. In that case, the particular parametrization of γ is not important. That is, for an increasing C^1 function ϕ and for $\tilde{\gamma} := \gamma \circ \phi$, we have

$$\int_\gamma f(z)dz = \int_{\tilde{\gamma}} f(z)dz$$

Next, we defined complex differentiability for a function $f : U \to \mathbf{C}$ at a point $z_0 \in U$ by the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

if this limit exists. We denote the limit by $f'(z_0)$. It turns out that this is equivalent with the function f being holomorphic. We proved the following theorem.

Theorem 1 $f \in C^1(U)$ is holomorphic if and only if $f'(z)$ exists for all $z \in U$. In this case, $f' = \frac{\partial f}{\partial z}$.

Finally, we extended the antiderivative condition as follows.

Proposition 2 Let P be a point in $\Omega \subset \mathbf{C}$, a simply connected open set. Suppose that F is holomorphic in $\Omega \backslash P$ and continuous on Ω. Then there exists H, holomorphic on Ω, such that $H'(z) = F$.

Week 3, February 3-7 We spent this week discussing and proving Cauchy integral formula and Cauchy integral theorem. Denote the integral over a closed curve γ by $\oint_\gamma$ and assume that the positive direction is counterclockwise. We have:

Cauchy Theorem Let Ω be simply connected domain and f is holomorphic on Ω. Then, for any closed C^1 curve $\gamma : [0,1] \to \Omega$ we have

$$\oint_\gamma f(z)dz = 0.$$

Cauchy integral formula Let Ω be a simply connected domain in $\mathbf{C}$ and f a holomorphic function on it. Let γ be a C^1 closed curve in Ω, which winds once around $z \in \Omega$. Then,

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\xi)}{\xi - z} d\xi.$$

We started the proof of this by computing the complex integral

$$\oint_{|\xi - z_0| = r} \frac{1}{\xi - z} = 2\pi i$$

whenever $z : |z - z_0| = r$. We then proved the results for the special case when γ is the boundary of a disc, and extended it to a general closed curve.

Monday, February 10 We worked on applications of Cauchy integral formula and theorem.

Theorem Let U be an open subset of $\mathbf{C}$ and f a holomorphic function on U. Then $f \in C^\infty(U)$ and for every integer k and every curve γ around $z \in U$

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_\gamma \frac{f(\xi)}{(\xi - z)^{k+1}} d\xi$$

As a corollary, we saw that if f is a holomorphic function on U, then f' is also holomorphic on U.

The next theorem shows that the Cauchy theorem is reversible.

Theorem (Morera) Let Ω be an open connected subset of $\mathbf{C}$ and f continuous function on U. Assume that for every closed C^1 curve γ, we have

$$\oint_\gamma f(z)dz = 0$$

Then f is holomorphic inside Ω. Finally, we discussed "holomorphic extension".

Proposition Let U be an open subset of $\mathbf{C}$ and f a holomorphic function on U. Let γ be a C^1 closed curve in U and $\phi \in C(\gamma)$. For $z \in Int(\gamma)$, define

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{\phi(\xi)}{\xi - z} d\xi$$

Then f is holomorphic inside γ.

Wednesday, February 12 During this and the next class, we focused on complex power series. We first proved Abel's lemma, which states that if $\sum_{k=0}^{\infty} a_k(z-P)^k$ converges at some z, then the series converges at each $\omega \in D(P,r)$, where $r = |z-P|$.

This allows to define the radius of convergence of a power series as

$$r := \sup\{|\omega - P| : \sum a_k(\omega - P)^k converges\}.$$

We proved the root test for the radius of convergence.
Lemma (root test)

$a) r = \frac{1}{\lim sup_{k\to+\infty}|a_k|^{1/k}}$ if $\lim sup_{k\to+\infty}|a_k|^{1/k} > 0$, or

$b) +\infty$ if $\lim sup_{k\to+\infty}|a_k|^{1/k} = 0$.

Next proposition states that inside the disc of convergence, the power series converges uniformly and absolutely.

Proposition Let $\sum_{k=0}^{\infty} a_k(z-P)^k$ be a power series with radius of convergence r. Then, for any number R with $0 \leq R < r$, the series converges uniformly and absolutely on $\bar{D}(P,R)$.

We also proved that the series obtained by term by term differentiation of the power series of a holomorphic function f is convergent and equal to the corresponding derivative. Finally, we established uniqueness of the power series, i.e. if

$$\sum_{j=0}^{\infty} a_j(z-P)^j = \sum_{j=0}^{\infty} b_j(z-P)^j$$

and both series are convergent on $D(P,r)$, then $a_j = b_j$ for every j.

Friday, February 14 A complex function is analytic if there exists a power series $f(z) = \sum_{j=0}^{\infty} a_j(z-P)^j$ for all $z : |z-P| < r$. We proved that holomorphic functions are analytic functions and the converse is also true.
Theorem If f is a holomorphic function on the set $U \subset \mathbf{C}$ and $P \in U, D(P,r) \subset U$. Then

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(P)}{j!}(z-P)^j$$

for all $z : |z-P| < r$. The radius of convergence at each point is at lest equal to the distance between P and ∂U.

Monday, February 17 We discussed Cauchy estimates and corollaries, including the Liouville's Theorem and the fundamental theorem of algebra.

Theorem Let f be a holomorphic function on an open set $U \subset \mathbf{C}$. Let $P \in U$ and $r > 0$ be such that $\bar{D}(P,r) \subset U$. Set $M = \sup_{z\in\bar{D}(P,r)} |f(z)|$. Then, for each $k = 1, 2, \ldots$ we have the estimate

$$|f^{(k)}(P)| \leq \frac{Mk!}{r^k}.$$

We say that a function $f\mathbf{C} \to \mathbf{C}$ is entire, if it is holomorphic on the whole $\mathbf{C}$.

Theorem (Liouville) An entire bounded function is a constant.

Corollary Assume that $f : \mathbf{C} \to \mathbf{C}$ is an entire function and there exist a real number C and a positive integer k such that $|f(z)| \leq |z|^k$ for all z with $|z| > 1$. Then $f(z)$ is a polynomial of degree at most k.

Finally, we used Liouville's theorem to give a proof of the fundamental theorem of algebra.

Corollary If $p(z)$ is a holomorphic polynomial of degree k, then there are k complex numbers $\alpha_1, \alpha_2, \ldots \alpha_k$ (not necessarily distinct) and a nonzero constant C such that

$$p(z) = C.(z - \alpha_1).(z - \alpha_2) \ldots (z - \alpha_k).$$

Wednesday, February 19 We studied the uniform limits of holomorphic functions and proved the following theorem.

Theorem Let U be an open set in $\mathbf{C}$. Let $\{f_j\}$ be a family of holomorphic functions on U, which converges uniformly over the compact subsets to f. Then f is holomorphic on U.

Thus, the set of holomorphic functions is closed under the operation uniform convergence over the compact subsets. Moreover, we have the following corollary.

Corollary Let U be an open set in $\mathbf{C}$. Let $\{f_j\}$ be a family of holomorphic functions on U, which converges uniformly over the compact subsets to f. Then for each positive integer k we have that $\{f_j^{(k)}\}$ converges uniformly over the compact subsets to $f^{(k)}$.

Friday, February 21 We studied the zeros of a holomorphic function f and showed that if $f \neq const$, then it can not have too many zeros.

Theorem Let U be an open and connected subset of $\mathbf{C}$ and $f : U \to \mathbf{C}$ be holomorphic. Then the set of zeros $Z = \{z \in U : f(z) = 0\}$ does not have accumulation points inside of U unless $f(z) = 0$. Equivalently, if there exist a sequence $\{z_j\}_{j=1}^{\infty}$ and $z_0 \in Z$ such that $z_j \to z_0$, then $f = 0$.

This does not exclude the possibility that for a non-trivial holomorphic function there is an accumulation point z_0 on ∂U. For example, the function $f(z) = \sin(\frac{1}{1-z})$ has zeros $z_n = 1 - \frac{1}{n\pi}$, which accumulate at $1 \in \partial D(0, 1)$.

We also proved a number of useful corollaries. Here are some of them.

Corollary Let f be holomorphic on an open and connected set U, such that $f_{|D(p,r)} = 0$ for some $P \in U, r > 0$. Then $f = 0$ on U.

Corollary Let U be an open and connected set and the functions f, g are holomorphic on U and such that $fg = 0$ on U. Then either $f = 0$ or $g = 0$ on U.

Monday and Wednesday, February 24 and 26 We started looking at functions holomorphic in a punctured disc $D(P, r) \setminus \{P\}$. If such a holomorphic function f has an isolated singularity at P, then there are three distinct possibilities:

(1) f is bounded in a neighborhood of P, that is there is $M > 0$ and $r > 0$, so that $D(P, r) \subset U$ and

$$\sup_{z \in D(P,R) \setminus \{P\}} \leq M.$$

(2) $\lim_{z \to P} |f(z)| = \infty$

(3) neither 1. or 2. holds. This is called an essential singularity.

We proved the following theorem about the first case, which is usually referred to as removable singularity.

Theorem (Riemann removable singularity) Suppose f is a holomorphic function in a punctured neighborhood of P and f is bounded in a neighborhood of P. Then $\lim_{z\to P} f(z)$ exists and the function

$$\tilde{f}(z) = \{ \begin{array}{cc} f(z) & z \neq P \\ \lim_{z\to P} f(z) & z = P \end{array}$$

is holomorphic.

Thus the function f admits a holomorphic extension on the whole domain.

We also proved a result about the case of essential singularity.

Theorem (Casorati-Weierstrass) Let f be holomorphic in $D(P, r_0) \setminus \{P\}$ and let P be an essential singularity for f. Then, for each $r : 0 < r < r_0,\ f(D(P,r)\setminus\{P\})$ is dense in $\mathbf{C}$.

You can work on problems 2, 3, 5 on page 145 and 8 a) and 9 on page 146.

Wednesday and Friday, March 4th and 6th We studied the behavior and the expansion of a function near a singular point. We introduced the notion of a Laurent series and proved the following lemma.

Lemma Let $\sum_{j=-\infty}^{+\infty} a_j(z-P)^j$ be a doubly infinite series that converges at one point (at least). Then there are unique nonnegative numbers r_1 and r_2 such that the series converges absolutely for all z with $r_1 < |z-P| < r_2$ and diverges for z with $|z-P| < r_1$ and $|z-P| > r_2$. Also, if $r_1 < r_1' \leq r_2' < r_2$, then $\sum_{j=-\infty}^{+\infty} a_j(z-P)^j$ converges absolutely and uniformly there.

Our ultimate goal is to prove that every holomorphic function on an annulus is given by a convergent Laurent series. We will postpone this for now, but we were able to prove the following uniqueness result.

Proposition If the Laurent series $\sum_{j=-\infty}^{+\infty} a_j(z-P)^j$ converges on an annulus $0 \leq r_1 < r_2 \leq \infty$ to a function f, then for any r satisfying $r_1 < r < r_2$, and each $j \in Z$ we have

$$a_j = \frac{1}{2\pi i} \oint_{|\xi - P| = r} \frac{f(\xi)}{(\xi - P)^{j+1}} d\xi$$

In particular, the a_j's are uniquely determined by f.

Monday, March 23 After a brief review of isolated singularities and the expansion near a singular point, we will begin the discussion of how to prove the existence of Laurent series expansions, which is done in section 4.3 in our book. The first main result is the Cauchy integral formula for an annulus. We are using the generalized notion of annulus as before, for $0 \leq r_1 < r_2 \leq +\infty$ and we assume that the function $f : D(P, r_2) \setminus \bar{D}(P, r_1) \to \mathbf{C}$ is holomorphic. Then we have:

Theorem With the above assumptions and for each s_1, s_2 such that $r_1 < s_1 < s_2 < r_2$ and each point $z \in D(P, s_1) \setminus \bar{D}(P, s_1)$ we have the Cauchy formula

$$f(z) = \frac{1}{2\pi i} \oint_{|\xi - P| = s_2} \frac{f(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i} \oint_{|\xi - P| = s_1} \frac{f(\xi)}{\xi - z} d\xi$$

We will go over the proof in details during class. This theorem gives almost immediately the following desired result about existence of Laurent expansion.

Theorem With the assumptions above, there exist complex numbers a_j such that

$$\sum_{j=-\infty}^{+\infty} a_j(z-P)^j$$

converges on $D(P,r_2)\setminus\bar{D}(P,r_1)$ to f. If $r_1<s_1<s_2<r_2$, then the series converges absolutely and uniformly on $D(P,r_2)\setminus\bar{D}(P,r_1)$.

We will continue on Wednesday with the special case when f is a holomorphic function in $D(P,r)$ with an isolated singularity at P.

Wednesday, March 25 For today's lecture, we will focus on the situation when f is a holomorphic function in $D(P,r)$ with an isolated singularity at P. In this situation, the existence of Laurent expansion and the formula for the coefficients remains valid, but we can discuss directly the three mutually exclusive possibilities of (1) a removable singularity at P, (2) a pole at P and (3) an essential singularity at P.

We will prove that (1) $\Leftrightarrow a_j=0$ for all $j<0$, (2) $\Leftrightarrow$ for some $k>0$, $a_j=0$ for all $\infty<j<-k$ and (3) is when neither (1) nor (2) applies.

Also, if a function f has a Laurent expansion $f(z)=\sum_{j=-k}^{\infty}a_j(z-P)^j$ for some $k>0$ and if $a_{-k}\neq 0$, then **f has a pole of order k at P**. Moreover, in that case $(z-P)^kf(z)$ is bounded near P, but $(z-P)^{k-1}f(z)$ is not.

Next, in section 4.4, we give an algorithm for calculating the coefficients of the Laurent expansion - the coefficients are given by

$$a_j=\frac{1}{(k+j)!}(\frac{\partial}{\partial z})^{k+j}((z-P)^kf)_{|z=P}$$

Finally, we work out several concrete examples of Laurent series expansions.

Monday, March 29 This week we will focus on functions with more than one isolated singularities, i.e. holomorphic on an open set with finitely many points removed. We require the domain U to be **holomorphically simply connected**, which means that the domain U is connected, and for each holomorphic function $f:U\to\mathbf{C}$ there exists a holomorphic antiderivative $F'=f$. From what we learned before, this is equivalent with the fact that for each piecewise C^1 closed curve γ in U, we have that $\oint_\gamma f=0$. Clearly discs, squares and $\mathbf{C}$ are holomorphically simply connected domains, but for example $D(0,1)\setminus\{0\}$ is not since the functions $f(z)=\frac{1}{z}$ has no holomorphic antiderivative on this set.

Our main goal is to prove the following classical residue theorem:

Theorem Suppose U is holomorphically simply connected open set in $\mathbf{C}$ and $\{P_1,P_2,\dots P_n\}$ are distinct points of U. Suppose also that $f:U\setminus\{P_1,P_2,\dots P_n\}\to\mathbf{C}$ is a holomorphic function and that γ is a closed piecewise C^1-curve in $U\setminus\{P_1,P_2,\dots P_n\}$. Set the residue of f at P_j to be R_j, the coefficient in front of $(z-P_j)^{-1}$ in the Laurent expansion of f about P_j. Then we have the following:

$$\oint_\gamma f=\sum_{j=1}^{n}R_j(\oint_\gamma\frac{1}{\xi-P_j}d\xi)$$

Using the winding number of the curve γ about the point P notion (index of γ with respect to P), defined as $Ind_\gamma(P) = \frac{1}{2\pi i}\oint_\gamma \frac{1}{\xi - P}d\xi$, we can formulate the theorem above as follows:

$$\oint_\gamma = 2\pi i \sum_{j=1}^{n} Res_f(P_j) Ind_\gamma(P_j)$$

Finally, we discuss a quick and easy way to compute residues using the formula $Res_f(P) = \frac{1}{(k-1)!}(\frac{\partial}{\partial z})^{k-1}((z-P)^k f(z))_{|z=P}$ for a pole of order k at P.

Wednesday, April 1 Through several examples, we present a collection of techniques that use the calculus of residues to compute indefinite integrals. Each example has specifics that are important and in most cases the integrals are harder or impossible to compute using regular calculus techniques.

Friday, April 3rd We will work on examples of different problems from Chapter 4 and will discuss questions and hints for the problem set in Homework 4.

Monday, April 6 We will discuss general results that allow us to count the number of zeros for holomorphic and meromorphic functions, which are in section 5.1 of the textbook.

Definition We say that a zero z_0 of a holomorphic function $f(z)$ is of multiplicity $k,\ k \in N$, if $f(z) = (z - z_0)^k g(z)$, where g is also holomorphic in a neighborhood of z_0 and $g(z_0) \neq 0$.

We will discuss in details and prove the following argument principle for holomorphic functions. Assume $U \subset \mathbf{C}$ is open and $f(z)$ is holomorphic on U. Suppose $\bar{D}(P,r) \subset U$ is such that $f_{|\partial D(P,r)} \neq 0$.

Proposition If $z_1, z_2, \ldots z_k$ are the zeros of f in the interior of the disc and let n_l be the order of the zero at z_l. Then

$$\frac{1}{2\pi i}\oint_{|\xi - P| = r} \frac{f'(\xi)}{f(\xi)} d\xi = n_1 + n_2 + \ldots n_k$$

This principle can be extended to yield information about meromorphic functions as follows:

Theorem (Argument principle for meromorphic functions) Assume $U \subset \mathbf{C}$ is open and $f(z)$ is meromorphic on U. Suppose $\bar{D}(P,r) \subset U$ is such that $f_{|\partial D(P,r)}$ has neither poles nor zeros. Then

$$\frac{1}{2\pi i}\oint_{|\xi - P| = r} \frac{f'(\xi)}{f(\xi)} d\xi = n_1 + n_2 + \ldots n_p - (m_1 + m_2 + \ldots m_q),$$

where $n_1, n_2, \ldots, n_p$ are the multiplicities of the zeros $z_1, z_2, \ldots, z_p$ and $m_1, m_2, \ldots, m_q$ are the orders of the poles $w_1, w_2, \ldots w_q$ of f in $D(p,r)$.

Wednesday, April 8 We discussed section 5.2, which deals with the local geometry of holomorphic functions. The main subject was the classical open mapping theorem. Functions, for which the direct image of any open set is open are called "open mappings".

Theorem If f is a non-constant holomorphic function on an open connected set U, then $f(U)$ is an open set in $\mathbf{C}$.

Friday, April 10 We discussed the solutions to several problems from Homework set 4.

Monday, April 13 We continue with the zero counting results, this time we want to be able to count the zeros of a holomorphic function on its whole domain of definition. The first result is the following.

Rouche's Theorem Suppose that f, g are holomorphic functions on an open set $U \subset \mathbf{C}$ and that $\bar{D}(P, r) \subset U$. Suppose that for each $\xi \in \partial D(P, r)$, one has

$$|f(\xi) - g(\xi)| < |f(\xi)| + |g(\xi)|$$

Then

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f'(\xi)}{f(\xi)} d\xi = \frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{g'(\xi)}{g(\xi)} d\xi$$

This means that the number of zeros of f in $D(P, r)$ counting multiplicities is equal to the number of zeros of g in $D(p, r)$ counting multiplicities.

Another useful consequence of the argument principle is the Hurwitz Theorem about the limit of a sequence of zero-free functions.

Theorem Let $U \subset \mathbf{C}$ be a connected open set and consider a family of holomorphic functions $f_j : U \to \mathbf{C}$, each nowhere vanishing on U. If the sequence f_j converges on the compact subsets of U to f, then f is either identically zero or vanishes nowhere on U.

Next, we discuss the so called maximum modulus principle and maximum modulus theorem (section 5.4).

Theorem (The maximum modulus principle) Let f be a holomorphic function on a domain U, $U \subset \mathbf{C}$. If there is a point $P \in U$ such that $|f(P)| \geq |f(z)|$ for all $z \in U$, then f is a constant.

Corollary (Maximum modulus theorem) If u is a bounded domain in $\mathbf{C}$ and f is a continuous function on $\bar{U}$ that is holomorphic on U, then the maximum value of $|f|$ on $\bar{U}$ must occur on the boundary ∂U.

Wednesday, April 15 We present the classical estimates of holomorphic functions on the unit disc that go by the name of Schwarz lemma and its generalizations.

Theorem (Schwarz lemma) Let f be holomorphic on the unit disc. Assume that
1. $f(z) \leq 1, \quad z \in D(0, 1)$
2. $f(0) = 0$

Then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Moreover, if either $|f(z_0)| = |z_0|$ or $|f'(0)| = 1$, then there exists $\alpha :\ |\alpha| = 1$ such that $f(z) = \alpha z$ (f is a rotation). There is also the following generalization of this theorem.

Theorem (Schwarz-Pick) Let f be holomorphic function on the unit disc with $|f(z)| \leq 1$ for all $z \in D(0, 1)$. Then for any $a \in D(0, 1)$ and with $b = f(a)$, we have the estimate

$$|f'(a)| \leq \frac{1 - |b|^2}{1 - |a|^2}$$

Moreover, if $f(a_1) = b_1$ and $f(a_2) = b_2$, then

$$|\frac{b_2 - b_1}{1 - \bar{b_1} b_2}| \leq |\frac{a_2 - a_1}{1 - \bar{a_1} a_2}|$$

Monday, April 20 In the last few weeks we will focus on infinite series and products and their applications, as developed in Chapters 8 and 9 in the book. We start with the basic concepts, defining rigorously the notion of convergence of infinite products.

Definition We say that the infinite product $\prod_{j=1}^{\infty}(a + a_j)$ converges, if

1. Only finitely many a_j's are equal to -1.
2. For N_0 large so that for all $j > N_0$, we have $a_j \neq -1$, we require that

$$\lim_{N\to\infty} \prod_{j=N_0+1}^{N} (1 + a_j)$$

converges and **the limit is not zero**.

We then define the value of the convergent product as

$$[\prod_{j=1}^{N_0}(1 + a_j)] \lim_{N\to+\infty} \prod_{N_0+1}^{N} (1 + a_j)$$

Note that if the infinite product $\prod_{j=1}^{\infty}(1+a_j)$ converges, then $\lim_{N\to+\infty} \prod_{j=1}^{N}(1+a_j)$ exists and equals the value of the infinite product, but the converse is not true.

Proposition The series $\sum_n |a_n|$ converges if and only if $\prod_{n=1}^{\infty}(1+|a_n|)$ converges.

We say that the product $\prod_{n=1}^{\infty}(1 + a_n)$ converges absolutely if $\prod_{n=1}^{\infty}(1 + |a_n|)$ converges.

Theorem Absolute convergence implies convergence for products. That is, if the product $\prod_{n=1}^{\infty}(1 + |a_n|)$ converges, then the product $\prod_{n=1}^{\infty}(1 + a_n)$ converges. In particular, if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\prod_{n=1}^{\infty}(1 + a_n)$ converges.

Finally, we can apply these results to infinite products of holomorphic functions.

Theorem Suppose $f_j : U \to \mathbf{C}$ are holomorphic, where $U \subset \mathbf{C}$ is open and that $\sum_{j=1}^{\infty} |f_j|$ converges uniformly on compact sets. Then the sequence of partial products

$$F_N(z) = \prod_{j=1}^{N}(1 + f_j(z))$$

converges uniformly on compact sets. In particular, the limit defines a holomorphic function F on U. This function vanishes at a point $z_0 \in U$ if and only if $f_j(z_0) = -1$ for some j. The multiplicity of the zero at z_0 is the sum of the multiplicities of the zeros of the functions $1 + f_j$ at z_0.

Friday, April 24 We proved a lemma and a very general theorem that lead to the formulation and the proof of the Weierstrass factorization theorem.

Lemma For the elementary Weierstrass factors, $E_p(z) = (1 - z)e^{(z+\frac{z^2}{2}+\ldots+\frac{z^p}{p})}$, we have for each $z : |z| \leq 1$ that $|E_p(z) - 1| \leq |z|^{p+1}$.

In other words, $E_p(z)$ approximates 1 well.

Using this lemma, as well as several results about infinite products, we can prove the following.

Theorem Let $\{a_n\}$ be a sequence of non-zero complex numbers without accumulation point. Suppose that p_n are integers, such that for all $r > 0$, there is

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|} \right)^{p_n+1} < \infty.$$

Then the infinite product

$$\prod_{n=1}^{\infty} E_{p_n} \left(\frac{z}{a_n} \right)$$

converges uniformly on the compact subsets of $\mathbf{C}$ to an entire function F. Moreover, the zeros of F are exactly the sequence $\{a_n\}$.

Corollary Let $\{a_n\}$ be a sequence of non-zero complex numbers without accumulation point. There exists an entire function f, such that f has exactly these zeros. Assuming that 0 is a zero of order m, we can take

$$f(z) = z^m \prod_{n=m+1}^{\infty} E_{n-1} \left(\frac{z}{a_n} \right).$$

Finally, we can state and prove the following Weierstrass factorization theorem.

Theorem Let the entire function f vanishes to order m at zero. Suppose $\{a_n\}$ are the other zeros of f, listed with their multiplicities. Then there exists an entire function g, such that

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{n-1} \left(\frac{z}{a_n} \right)$$

Monday and Wednesday, April 27 and 29 We discussed Blaschke products and Jensen's formula, which are the main applications for the infinite product concepts developed last week.

Proposition The Blaschke factor $B_a(z) = \frac{z-a}{1-\bar{a}z}$, defined for $|a| < 1$ is a holomorphic function on a neighborhood of $\bar{D}(0,1)$. Also, $B_a(a) = 0, B_a(z) \neq 0$ for $z \neq a$ and $|B_a(z)| = 1$ if $|z| = 1$.

Theorem (Jensen's formula) Let f be a holomorphic function on a neighborhood of $\bar{D}(0,r)$ with $f(0) \neq 0$, also let $a_1, a_2, \ldots, a_k$ be the zeros of f in $\bar{D}(0,r)$, counted with multiplicities. Then

$$\log|f(0)| + \sum_{j=1}^{k} \log|\frac{r}{a_j}| = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{1\theta})| d\theta$$

Jensen's inequality follows immediately from this theorem.

Corollary With f as in the theorem above, we have the inequality

$$\log|f(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{1\theta})| d\theta$$

Next, we prove the following theorem.

Theorem If f is a nonconstant bounded holomorphic function on $D(0,1)$ and $a_1, a_2, \ldots$ are the zeros of f, counted with multiplicities, then

$$\sum_{j=1}^{\infty}(1-|a_j|) < \infty.$$

Moreover, the reverse if also trua as the next result shows.

Theorem If $\{a_j\} \subset D(0,1)$ satisfies $\sum_{j=1}^{\infty}(1-|a_j|) < \infty$ and no $a_j = 0$, then there is a bounded holomorphic function on $D(0,1)$, whose zeros coincide with the $\{a_j\}$. More precisely, the infinite product

$$\prod_{j=1}^{\infty} \frac{-\bar{a}_j}{|a_j|} B_{a_j}(z)$$

converges uniformly on the compact subsets of $D(0,1)$ to a bounded holomorphic function $B(z)$. $B(z)$ has zeros at precisely the a_j's, counted with multiplicities.

Definition An expression of the form

$$z^m \prod_{j=1}^{\infty} \frac{-\bar{a}_j}{|a_j|} B_{a_j}(z),$$

where $m > 0$ is an integer, is called Blaschke product.

Finally, we have the following corollary.

Corollary Suppose that f is a bounded holomorphic function on $D(0,1)$, vanishing to order $M \geq 0$ at 0 and that $\{a_j\}$ are its other zeros listed with multiplicities. Then

$$f(z) = z^m \prod_{j=1}^{\infty} \frac{-\bar{a}_j}{|a_j|} B_{a_j}(z) F(z),$$

where F is a bounded holomorphic function on $D(0,1), F$ has no zeros and

$$\sup_{z\in D(0,1)} |f(z)| = \sup_{z\in D(0,1)} |F(z)|.$$